Representability. Fix a category $\mathcal{C}$ and a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$. Yoneda's lemma states that for $c \in \mathcal{C}$, the set map $よ(c, F)\colon \operatorname{Hom}(\mathcal{C}(- \to c) \Rightarrow F) \to F(c)$ given by $\alpha \mapsto \alpha_c(\mathrm{id}_c)$ is bijective and natural in both c and F. Its inverse sends an element $x \in F(c)$ to the natural transformation $\mathrm{ev}_x\colon \mathcal{C}(- \to c) \Rightarrow F$ whose component map at $c' \in \mathcal{C}$ is the evaluation $(\mathrm{ev}_x)_f\colon \mathcal{C}(c' \to c) \ni f \mapsto F(f)(x)$. The *Yoneda embedding* of $\mathcal{C}$ is the fully faithful functor $よ\colon \mathcal{C} \to \mathsf{Fun}(\mathcal{C}^{\mathrm{op}} \to \mathsf{Set})$ given on objects by $c \mapsto \mathcal{C}(- \to c)$ and on morphisms by $(f\colon a \to b) \mapsto f_*\colon \mathcal{C}(- \to a) \to \mathcal{C}(- \to b)$.

We call F *representable* if there is a natural isomorphism $\alpha\colon F \Rightarrow \mathcal{C}(- \to c)$ for some $c \in \mathcal{C}$. In this case we call c the *representing object* and (c, α) the *representing pair* for F. The category of representing pairs for F is contractible, i.e, there is a unique representing pair (c, α) for F up to a unique structure-preserving isomorphism.

For $a \in \mathcal{C}$ and $u \in F(a)$, the pair (a, u) is called *universal* if it has the following *universal property*: for every $b \in \mathcal{C}$ and $v \in F(b)$, there is a unique morphism $f_b\colon b \to a$ such that $F(f_b)(u) = v$. An equivalent condition is that for every $b \in \mathcal{C}$, morphisms $f_b\colon b \to a$ bijectively correspond to elements of $F(b)$ via the map $f_b \mapsto F(f_b)(u)$. We call a a *universal object* for F.

Theorem. *Consider a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$, an object $a \in \mathcal{C}$, and an element $u \in F(a)$. If (a, α) is a representing pair for F, then $\eta_a(\mathrm{id}_a)$ is a universal object for F. Conversely, if (a, u) is universal for F, then the natural isomorphism $\mathcal{C}(- \to a) \Rightarrow F$ sending a morphism $f \in \mathcal{C}(b \to a)$ to the element $F(f)(u) \in F(b)$ makes (a, α) a representing pair for F.*

Thus a representable functor corresponds to a universal property characterizing morphisms out of a universal object. It is common that F maps objects of $\mathcal{C}$ to sets whose elements are required to make certain diagrams commute. Then the universal property is the usual notion. As an exercise, determine which functor is represented by the pullback of sets.