Contents

1 Homework 1

1.1 Exercise: Folland Exercise 5.29.

Let $Y = L^1(\mu)$ where μ is the counting measure on $\mathbb{Z}_{\geqslant 1}$, and let $X = \{f \in Y \mid \sum_1^\infty n|f(n)| < \infty\}$, equipped with the L^1 norm.

(a) X is a proper dense subspace of Y; hence X is not complete.

(b) Define $T\colon X \to Y$ by $Tf(n) := nf(n)$. Then T is closed but not bounded.

(c) Let $S := T^{-1}$. Then $S\colon Y \to X$ is bounded and surjective but not open.

Solution. Let $\mathbb{K}$ denote $\mathbb{R}$ or $\mathbb{C}$. As μ is the counting measure on $\mathbb{Z}_{\geqslant 1}$, we can make the identifications

$$Y = \left\{\{a_n\} \;\middle|\; a_n \in \mathbb{K} \text{ and } \sum\nolimits_1^\infty |a_n| < \infty\right\}$$

and

$$X = \left\{\{a_n\} \;\middle|\; a_n \in \mathbb{K} \text{ and } \sum\nolimits_1^\infty n|a_n| < \infty\right\}.$$

(a)

- X is properly contained in Y: First note X is contained in Y, since if $\sum_1^\infty n|a_n| < \infty$ then $\sum_1^\infty n|a_n| < \infty$. The containment is proper, since the sequence $a_n = 1/n^2$ has $\{a_n\}_{n=1}^\infty \in Y \setminus X$. Hence $X \subsetneq Y$.
- X is a linear subspace of Y: Let $\{a_n\}, \{b_n\} \in X$ and $\lambda \in \mathbb{K}$. Then for any $N \in \mathbb{Z}_{\geqslant 1}$,
$$\sum\nolimits_1^N n|a_n + \lambda b_n| \leqslant \sum\nolimits_1^N (n|a_n| + n|b_n|) + \sum\nolimits_1^N n|a_n| + \sum\nolimits_1^N n|b_n|.$$
Sending $n \to \infty$, we obtain
$$\sum\nolimits_1^\infty n|a_n + \lambda b_n| \leqslant \sum\nolimits_1^\infty n|a_n| + \sum\nolimits_1^\infty n|b_n| < \infty,$$

where the last inequality is because $\{a_n\}, \{b_n\} \in X$. Hence $\{\lambda a_n + b_n\} \in X$, so X is a linear subspace.

- X is dense in Y: Since simple functions are dense in $Y = L^1(\mu)$, it suffices to show X contains all simple functions in $L^1(\mu)$. So let $g = \{b_n\} \in L^1(\mu)$ be a simple function, that is, $g = \sum_1^N z_j \chi_{E_j}$ for finitely many $E_j \in \mathcal{P}(\mathbb{Z}_{\geqslant 1})$. Note that there exist at most finitely many $n \in \mathbb{Z}_{\geqslant 1}$ such that $b_n \neq 0$: indeed, if there exists $k \in \{1, \ldots, N\}$ such that both $z_j \neq 0$ and E_k is an infinite set, then

$$\infty = \sum_{\ell=1}^{\infty} c_k \mu(E_k) \leqslant \sum_{\ell=1}^{\infty} c_\ell(E_\ell) = \int g \, \mathrm{d}\mu,$$

contradicting $g \in L^1(\mu)$. Thus $\int g = \sum_{n=1}^{\infty} n|b_n|$ is a finite sum, and hence is finite. It follows that $g \in Y$, so Y is dense in X.

(b)
- T is not bounded: Fix an arbitrary $m \in \mathbb{Z}_{\geqslant 1}$ and define $f_m(n) = 1$ if $m = n$ and $f_m(n) = 0$ otherwise. Then $\sum_n n|f_m(n)| = n < \infty$, so $f_m \in X$. But $\|Tf_m\| = \sum_n n|Tf_m(n)| = \sum_n n^2|f_m(m)| = m^2 = m\|f_n\|$, so $\|T\|_{\text{op}} \leqslant m$. But m was an arbitrary nonnegative integer, so $\|T\|_{\text{op}} = \infty$. Hence T is not bounded.
- T is closed: Suppose $f(n) \to f$ in X and $Tf(n) \to g$ in Y. We claim $Tf = g$. First fix $\varepsilon > 0$. By our assumption, for all sufficiently large N we have $\sum_{n=N}^{\infty} n|f(n)| < \varepsilon/4$, $\sum_{n=N}^{\infty} |g(n)| < \varepsilon/4$, $\|g - Tf_n\| < \varepsilon/4$, and $\|f - f_n\| < \frac{\varepsilon}{4N}$. Then for all sufficiently large m and N, we have

$$\sum_{n=1}^{\infty} |Tf(n) - Tf_m(n)| = \sum_{n=1}^{N-1} |nf(n) - nf_m(n)| + \sum_{n=N}^{\infty} |nf(n) - Tf_m(n)|$$
$$< \sum_{n=1}^{N-1} |f(n) - f_m(n)| + \varepsilon/4 + \sum_{n=N}^{\infty} |Tf_m(n) - g(n)| + \sum_{n=N}^{\infty} |g(n)| < \varepsilon,$$

so $Tf_n \to Tf$ in L^1. Since $Tf(n) \to g$ by assumption, we conclude by uniqueness of limits in a normed (hence Hausdorff) vector space (namely, $L^1(\mu)$) that $Tf = g$.

(c) Fix $f \in Y$. Then $Sf(n) = n^{-1}f(n)$ for any $n \in \mathbb{Z}_{\geqslant 1}$, so

$$\|Sf\| = \sum_{n=1}^{\infty} n^{-1}|f(n)| \leqslant \sum_{n=1}^{\infty} |f(n)| = \|f\|.$$

Thus $\|S\|_{\text{op}} \leqslant 1$, so S is bounded. And S is surjective, since any $\{a_n\} \in X$ is the image under S of the sequence $\{\frac{a_n}{n}\}$ (since if $\sum n|a_n| < \infty$ then in particular $\sum \frac{1}{n}|a_n| < \infty$, meaning $\{\frac{a_n}{n}\} \in Y$). Lastly, if S were open, then $T = S^{-1}$ is continuous, which contradicts part (b). Thus S is not an open map, as claimed. □

1.2 Exercise: Folland Exercise 5.32.

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on the vector space X such that $\|\cdot\|_1 \leqslant \|\cdot\|_2$. If X is complete with respect to both norms, then the norms are equivalent.

Solution. The linear operator $T\colon (X, \|-\|_2) \to (X, \|-\|_1)$ defined by $Tx \coloneqq x$ is bounded, since by hypothesis $\|Tx\|_1 = \|x\|_1 \leqslant \|x\|_2$ for all $x \in X$. Since T is a bijection of sets, $T^{-1} \in L((X, \|-\|_1), (X, \|-\|_2))$ by the bounded inverse mapping theorem. Hence there exists $C_2 > 0$ such that $\|x\|_2 = \|T^{-1}x\|_2 \leqslant C\|x\|_1$. Thus

$$\|x\|_1 \leqslant \|x\|_2 \leqslant C\|x\|_1$$

for all $x \in X$, so $\|-\|_1$ and $\|-\|_2$ are equivalent. □

1.3 Exercise: Folland Exercise 5.37.

Let X and Y be Banach spaces. If $T\colon X \to Y$ is a linear map such that $f \circ T \in X^*$ for every $f \in Y^*$, then T is bounded.

Solution. Suppose $x_n \to x$ and $Tx_n \to y$. We claim $y = Tx$. On one hand, by continuity of f we have

$$\lim_{n\to\infty} f \circ T(x_n) = f\Big(\lim_{n\to\infty} Tx_n\Big) = f(y).$$

On the other hand, $f \circ T \in X^*$ by hypothesis, so in particular $f \circ T$ is continuous; hence

$$\lim_{n\to\infty} f \circ T(x_n) = f \circ T\Big(\lim_{n\to\infty} x_n\Big) = f \circ T(x).$$

Thus

$$f(y) = f \circ T(x) \text{ for all } y \in Y^*. \tag{1.1}$$

It follows that $y = Tx$, since otherwise there exists $f \in Y^*$ such that $f(y) \neq f(Tx)$ (since by a corollary to the Hahn-Banach theorem X^* separates points), contradicting Equation (1.1). It then follows that the graph of T is closed, so by the closed graph theorem T is bounded. □

1.4 Exercise: Folland Exercise 5.39.

Let X, Y, Z be Banach spaces and let $B\colon X \times Y \to Z$ be a ***separately continuous*** bilinear map; that is, $B(x, -) \in L(Y, Z)$ for each $x \in X$, and $B(-, y) \in L(X, Z)$ for each $y \in Y$. Then B is jointly continuous, that is, continuous from $X \times Y$ to Z. [1]

Solution. To show B is bounded as a linear map $X \times Y \to Z$, we need to show there exists a constant C such that $\|B(x, y)\|_Z = \|(x, y)\|_{X \times Y}$ for all $(x, y) \in X \times Y$.

If $X = Y = \{0\}$ then all bilinear maps $X \times Y \to Z$ are continuous, so we may assume one of X and Y is not $\{0\}$.

Observe that if $x = 0$ or $y = 0$ for some $(x, y) \in X \times Y$ such that $\|(x, y)\|_{X \times Y} = 1$, then $\|B(x, 0)\|_Z = \|0\|_Z = 0 \leqslant \|(x, y)\|_X = 1$. Thus

$$\|x\|_X \|y\|_Y \leqslant \|(x, y)\|_{X \times Y}.$$

for all $x \in X$ and all $y \in Y$, It then suffices to show there exists a constant C such that $\|B(x, y)\|_Z \leqslant C\|x\|_X \|y\|_Y$ for all $x \in X$ and all $y \in Y$. First note

$$\|B(x, y)\|_Z \leqslant \|B(-, y)\|_{\mathrm{op}} \|x\|_X. \tag{1.2}$$

Now consider the collection $\mathcal{A} = \{B(-, y) \mid y \in Y\}$. By hypothesis $\mathcal{A} \subset L(X, Z)$ and $\sup_{y \in Y} \|B(x, y)\|_Z < \infty$ for each fixed $x \in X$, so by the uniform boundedness principle

$$C \coloneqq \sup_{y \in Y} \|B(-, y)\|_{\mathrm{op}} < \infty.$$

We then conclude by Equation (1.2) that

$$\|B(x, y)\|_Z \leqslant C\|x\|_X \|y\|_Y.$$

□

1.5 Exercise.

Assume that T is a bounded linear map on $L^2([0,1])$ with the property that Tf is continuous on $[0,1]$ whenever f is continuous on $[0,1]$. Prove that the restriction of T to $C([0,1])$ is a bounded operator on $C([0,1])$, where as usual $C([0,1])$ is equipped with the uniform norm.

Solution. We will use the closed graph theorem. Suppose both $f_n \to f$ and $Tf_n \to g$ uniformly. We claim $Tf = g$. We first state and prove a useful lemma:

1.6 Lemma.

For all $f \in C([0,1])$ and all real numbers $p \in [1,\infty)$, $\|f\|_{L^p} \leqslant \|f\|_u$, where $\|-\|_u$ is the sup-norm.

Proof. Since $f \in C([0,1])$, $\|f\|_u$ is finite. Thus

$$\|f\|_{L^p}^p = \int_0^1 |f|^p \, dy \leqslant \int_0^1 \|f\|_u^p \, dy = \|f\|_u^p.$$

Taking the pth root of both sides, we obtain the desired inequality $\|f\|_{L^p} \leqslant \|f\|_u$. □

Since $T \in L(L^2([0,1]), L^2([0,1]))$, there exists $C > 0$ such that

$$\|Tf_n - Tf\|_{L^2} \leqslant C\|f_n - f\|_{L^2} \leqslant C\|f_n - f\|_u,$$

where the final inequality is by Lemma 1.6. Since $f_n \to f$ uniformly, it follows that $Tf_n \to Tf$ in $L^2([0,1])$. But also $Tf_n \to g$ uniformly by assumption, so in particular $Tf_n \to g$ in $L^2([0,1])$. And $L^2([0,1])$ is Hausdorff as a normed vector space, so by uniqueness of limits $Tf = g$. Thus, by the closed graph theorem, we conclude $T \in L(C([0,1]), C([0,1]))$. □

3 Homework 2

3.1 Exercise: Folland Exercise 6.7.

If $f \in L^p \cap L^\infty$ for some $p < \infty$, so that $f \in L^q$ for all $q > p$, then $\|f\|_\infty = \lim_{q\to\infty} \|f\|_q$.

Solution. First suppose $\|f\|_p = 0$. Then $0 = \|f\|_p^p = \int |f|^p$, so $|f| = 0$ a.e. This means $\|f\|_\infty = 0$ and $\|f\|_q = 0$ for all q, so

$$\|f\|_\infty = 0 = \lim_{q\to\infty} 0 = \lim_{q\to\infty} \|f\|_q,$$

which affirms the claim.

Now suppose $\|f\|_p > 0$. By Folland Proposition 6.10 with $r = \infty$, for all $q > 0$ and all $p \in (1, q)$ we have

$$\|f\|_q \leqslant \|f\|_p^{p/q} \|f\|_\infty^{1-p/q}.$$

Taking the limit at $q \to \infty$, we obtain

$$\lim_{q\to\infty} \|f\|_q \leqslant \|f\|_p^{p/q} \|f\|_\infty^{1-p/q} = \|f\|_p^0 \|f\|_\infty^{1-0} = \infty,$$

where we used that the map $q \mapsto \|f\|_p^q$ is continuous as a function of $q \in (0, \infty)$ (since $\|f\|_p$ is nonnegative).

To show the reverse inequality, it suffices to show $\liminf_{q\to\infty} \|f\|_q \leqslant \|f\|_\infty$. We can prove this as follows: Fix $n \in \mathbb{Z}_{\geqslant 1}$ and let

$$E_n := \{x \in X \mid |f| \geqslant \|f\|_\infty - 1/n\}.$$

Since $\mu(E_n) > 0$ (by definition of $\|-\|_\infty$), we have

$$\|f\|_q^q = \int |f|^q \geqslant \int_{E_n} |f|^q \geqslant \int_{E_n} (\|f\|_\infty - 1/n)^q = \mu(E_n)(\|f\|_\infty - 1/n)^q.$$

Taking the qth root of both sides, we obtain

$$\|f\|_q \geqslant \mu(E_n)^{1/q}(\|f\|_\infty - 1/n). \tag{3.1}$$

And $\mu(E_n) < \infty$, since otherwise $\infty = \mu(E_n)^{1/q}(\|f\|_\infty - 1/n) \leqslant \|f\|_q^q$, contradicting $f \in L^q$. Also $\mu(E_n) > 0$ (by definition of $\|-\|_\infty$), so by taking $q \to \infty$ we have by Equation (3.1) that

$$\lim_{q\to\infty} \|f\|_q \geqslant \mu(E_n)^0(\|f\|_\infty - 1/n) = \|f\|_\infty - 1/n.$$

Since n was arbitrary, we conclude $\lim_{q\to\infty} \|f\|_q \geqslant \|f\|_\infty$, which completes the proof. □

3.2 Exercise: Folland Exercise 6.8.

Suppose $\mu(X) = 1$ and $f \in L^p$ for some $p > 0$, so that $f \in L^q$ for $0 < q < p$.

(a) $\log \|f\|_q \geqslant \int \log |f|$. (Use Folland Exercise 3.42(d) with $F(t) = e^t$.)

(b) $\left(\int |f|^q - 1\right)/q \geqslant \log \|f\|_q$, and $\left(\int |f|^q - 1\right)/q \to \int \log |f|$ as $q \searrow 0$.

(c) $\lim_{q \searrow 0} \|f\|_q = \exp\left(\int \log |f|\right)$.

Solution.

(a) Here we use the convention $\log(0) = -\infty$ and $\log \infty = \infty$. We may assume $\int \log|f| \neq -\infty$, since otherwise the desired inequality is

$$\log|f|^q = q \int \log|f| = -\infty \leqslant \log\|f\|_q,$$

which holds irregardless of the value of $\|f\|_q$. The exponential is convex and $\mu(X) = 1$, so by Jensen's inequality (Folland Exercise 3.42(d)), we obtain

$$\exp\left(\int \log|f|^q\right) \leqslant \int \exp(\log|f|^q) = \int |f|^q.$$

Taking the logarithm of both sides, we deduce

$$q \int \log|f| = \int \log|f|^q \leqslant \log \int |f|^q = \log\|f\|_q^q = q \log\|f\|_q.$$

By dividing through by $q > 0$, we conclude $\int \log|f| \leqslant \log\|f\|_q$

(b) Since $\log x \leqslant x - 1$ for all $x \in [0, \infty]$, we have

$$q \log\|f\|_q = \log \int |f|^q \leqslant \int |f|^q - 1.$$

Then divide through by $q > 0$ to obtain the desired inequality.

It remains to show $\left(\int |f|^q - 1\right)/q \to \int \log|f|$ as $q \searrow 0$. We have $\chi_{\{|f| \geqslant 1\}} \frac{|f|^q - 1}{q} \leqslant \chi_{\{|f| \geqslant 1\}} \frac{|f|^p - 1}{p} \in L^1$, so by the dominated convergence theorem

$$\lim_{q \searrow 0} \int \chi_{\{|f| \geqslant 1\}} \frac{|f|^q - 1}{q} = \int \lim_{q \searrow 0} \chi_{\{|f| \geqslant 1\}} \frac{|f(x)|^q - 1}{q} = \int \chi_{\{|f| \geqslant 1\}} \log|f|, \tag{3.2}$$

where for the second equality we used the limit definition of the logarithm on $[0, \infty]$. On the other hand, by the fundamental theorem of calculus, we have

$$\chi_{\{|f| < 1\}} \frac{|f|^q - 1}{q} = \int_1^{|f|} \chi_{\{|f| < 1\}} t^{q-1} = \int_{|f|}^1 \chi_{\{|f| < 1\}} t^{q-1},$$

which increases as q decreases. As everything here is measurable, by the monotone convergence theorem

$$\lim_{q \searrow 0} \int \chi_{\{|f| < 1\}} \frac{|f|^q - 1}{q} = \int \chi_{\{|f| < 1\}} \log|f|. \tag{3.3}$$

Now by Equations (3.2) and (3.3), we conclude

$$\begin{aligned}\lim_{q\searrow 0}\int \frac{|f|^q-1}{q} &= \lim_{q\searrow 0}\int (\chi_{\{|f|<1\}}+\chi_{\{|f|\geqslant 1\}})\frac{|f|^q-1}{q} \\ &= \int \chi_{\{|f|<1\}}\log|f| + \int \chi_{\{|f|\geqslant 1\}}\log|f| = \int \log|f|,\end{aligned}$$

as claimed.

(c) We have

$$\exp\left(\int \log|f|\right) \leqslant \exp(\log\|f\|_q) \leqslant \exp\left(\int |f|^q - 1\right)/q,$$

where the first and second inequalities are by parts (a) and (b), respectively. By part (b) and continuity of the exponential,

$$\exp\left(\int |f|^q - 1\right)/q \to \int \log|f|$$

as $q \to 0$. Now by the squeeze theorem for limits, we conclude

$$\lim_{q\to 0}\|f\|_q = \exp\left(\int \log|f|\right). \qquad \square$$

3.3 Exercise: Folland Exercise 6.10.

Suppose $1 \leqslant p < \infty$. If $f_n, f \in L^p$ and $f_n \to f$ a.e., then $\|f_n - f\|_p \to 0$ if and only if $\|f_n\|_p \to \|f\|_p$. (Use Folland Exercise 2.20.) In addition, prove or disprove the assertion in the case $p = \infty$.

Solution.

($\Rightarrow$) If $\varepsilon > 0$ and $\|f_n - f\|_p \to 0$, then by the triangle inequality $\|f_n\|_p - \|f\|_p \leqslant \|f_n - f\|_\infty < \varepsilon$ for all sufficiently large $n \in \mathbb{Z}_{\geqslant 1}$, so the forward implication holds. Note that this argument works for all $p \in [1, \infty]$.

($\Leftarrow$) Since $\|f_n\|_p \to \|f\|_p$, we have $\|f_n\|_p^p \to \|f\|_p^p$. Setting $g_n := 2^p \max\{|f_n|^p, |f|^p\}$, $g := 2^p|f|^p \geqslant 0$, $h_n := 2^p|f_n - f|^p$, and $h := 0$, we observe that

- $h_n \to h$ a.e.,
- $g_n \to g$ a.e.,
- $g_n \in L^1$ since $f_n, f \in L^p$ implies $|f_n|^p, |f|^p \in L^1$ (hence also $\max\{|f_n|^p, |f|^p\} \in L^p$),
- $h_n \in L^1$ since by the triangle inequality $h_n \leqslant 2^p \max\{|f|_n^p, |f|^p\} = g_n \in L^1$ and g_n,
- $|h_n| = |f_n - f|^p \leqslant (|f_n| + |f|)^p \leqslant 2\max\{|f_n|^p, |f|^p\} \leqslant 2^p \max\{|f_n|^p, |f|^p\} = g_n \in L^1$ (since $f_n, f \in L^p$, hence $|f_n|^p, |f|^p \in L^1$), and
- $\int g_n = 2^p \int \max\{|f_n|^p, |f|^p\} \to 2^p \int |f|^p = \int g$ by hypothesis.

We can therefore apply the generalized dominated convergence theorem (Folland Exercise 2.20) to obtain

$$2^p \int |f_n - f|^p = \int h_n \to \int h = \int 0 = 0.$$

By dividing through by $2^p > 0$, we obtain

$$\|f_n - f\|_p^p \to 0,$$

which implies $\|f_n - f\|_p \to 0$.

The above argument fails in the case $p = \infty$: if $p = \infty$, then when the measure space is $(R, \mathcal{L}, m)$, we have

$$\left|\|\chi_{(-n,n)}\|_\infty - \|\chi_\mathbb{R}\|_\infty\right| = 0 \to 0 \text{ as } n \to \infty,$$

but

$$\|\chi_{(-n,n)} - \chi_\mathbb{R}\|_\infty = 1 \not\to 0 \text{ as } n \to \infty.$$

□

3.4 Exercise: Folland Exercise 6.12.

Show that for all $p \in [1,\infty] \setminus \{2\}$, the L^p norm does not arise from an inner product on L^p, except in trivial cases when $\dim(L^p) \leqslant 1$. (Show that the parallelogram law fails.)

Solution. Let $(X, \mathcal{M}, \mu)$ be a measure space. Recall that since $\dim L^p \geqslant 2$, there exist disjoint sets $A, B \in \mathcal{M}$ of positive finite measure. Then for all $p \in [1,\infty) \setminus \{2\}$,

$$\begin{aligned}
2\left\|\frac{\chi_A}{\mu(A)^{1/p}}\right\|_p + 2\left\|\frac{\chi_B}{\mu(B)^{1/p}}\right\|_p &= 4 \neq 4^{1/p} + 4^{1/p} = (1+1)^{2/p} + (1+1)^{2/p} \qquad \text{(since } p \neq 2\text{)} \\
&= \left(\frac{1}{\mu(A)}\int|\chi_A|^p + \frac{1}{\mu(B)}\int|\chi_B|^p\right)^{2/p} + \left(\frac{1}{\mu(A)}\int|\chi_A|^p - \frac{1}{\mu(B)}\int|\chi_B|^p\right)^{2/p} \\
&= \left(\int\left|\frac{\chi_A}{\mu(A)^{1/p}}\right|^p + \int\left|\frac{\chi_B}{\mu(B)^{1/p}}\right|^p\right)^{2/p} + \left(\int\left|\frac{\chi_A}{\mu(A)^{1/p}}\right|^p - \int\left|\frac{\chi_B}{\mu(B)^{1/p}}\right|^p\right)^{2/p} \\
&= \left(\int\left|\frac{\chi_A}{\mu(A)^{1/p}} + \frac{\chi_B}{\mu(B)^{1/p}}\right|^p\right)^{2/p} + \left(\int\left|\frac{\chi_A}{\mu(A)^{1/p}} - \frac{\chi_B}{\mu(B)^{1/p}}\right|^p\right)^{2/p} \qquad \text{(since } A \cap B = \varnothing\text{)} \\
&= \left\|\frac{\chi_A}{\mu(A)^{1/p}} + \frac{\chi_B}{\mu(B)^{1/p}}\right\|_p^2 + \left\|\frac{\chi_A}{\mu(A)^{1/p}} - \frac{\chi_B}{\mu(B)^{1/p}}\right\|_p^2
\end{aligned}$$

Hence the parallelogram law fails. And if $p = \infty$, then with A and B as above we have

$$2 = \|\chi_A + \chi_B\|_\infty + \|\chi_A - \chi_B\|_\infty \neq 4 = 2\|\chi_A\|_\infty + 2\|\chi_B\|_\infty.$$

Thus for all $p \in [1,\infty] \setminus \{2\}$, $\|-\|_p$ does not arise from an inner product. □

3.5 Exercise.

Determine precisely the set of triples $(p, q, r) \in \overline{\mathbb{R}}^3$ with $1 \leqslant r \leqslant p, q \leqslant \infty$ such that the following holds: if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $fg \in L^r(\mathbb{R}^n)$ and $\|fg\|_r \leqslant \|f\|_p \|g\|_q$. (Here the underlying measure is Lebesgue measure.) Prove your answer.

Solution. We claim the set of triples for which this holds is given by

$$\mathcal{R} := \left\{(p, q, r) \in \overline{\mathbb{R}}^3 \;\middle|\; 1/p + 1/q = 1/r\right\}.$$

Proof. First suppose $(p, q, r) \in \mathcal{R}$, $f \in L^p(\mathbb{R}^n)$, and $g \in L^q(\mathbb{R}^n)$.

- Case 1: $1 \leqslant r \leqslant p, q < \infty$. Then $|f|^r \in L^{p/r}(\mathbb{R}^n)$ and $|g|^r \in L^{q/r}(\mathbb{R}^n)$, so by Hölder's inequality $|fg|^r = |f|^r |g|^r \in L^1(\mathbb{R}^n)$, hence $fg \in L^r$, and
$$\||fg|^r\|_1 \leqslant \||f|^r\|_{p/r} \||g|^r\|_{q/r}.$$
By raising both sides to the power of $1/r$, we obtain
$$\||fg|^r\|_1^{1/r} \leqslant \||f|^r\|_{p/r}^{1/r} \||g|^r\|_{q/r}^{1/r}, \tag{3.4}$$
so
$$\begin{aligned}\|fg\|_r = \left(\int |fg|^r\right)^{1/r} &= \||fg|^r\|_1^{1/r} \overset{(3.4)}{\leqslant} \||f|^r\|_{p/r}^{1/r} \||g|^r\|_{q/r}^{1/r} \\ &= \left(\int (|f|^r)^{p/r}\right)^{\frac{1}{\not r} \cdot \frac{\not r}{p}} \left(\int (|g|^r)^{q/r}\right)^{\frac{1}{\not r} \cdot \frac{\not r}{q}} = \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q} = \|f\|_p \|g\|_q.\end{aligned}$$
- Case 2: $1 \leqslant r \leqslant p < q = \infty$ or $1 \leqslant r \leqslant q < p = \infty$. (Without loss of generality take $1 \leqslant r \leqslant p < q = \infty$.) Then $1/r = 1/p$, and since $g \in L^\infty$, there exists a bounded function g' such that $g' = g$ a.e.; thus $|fg'|^p = |fg|^p$ a.e., so
$$\|fg\|_p^p = \|fg'\|_p^p = \int |fg'|^p \leqslant \|g'\|_\infty^p \int |f|^p = \|g'\|_\infty^p \|f\|_p^p < \infty.$$
Hence $fg \in L^r$ $(= L^p)$, and by taking the pth root of both sides (and noting that the right-hand side is just $\|g\|_\infty^p \|f\|_p^p$ since $g = g'$ a.e.), we recover the desired inequality.
- Case 3: $p = q = r = \infty$. Then the claim holds, since if $E \in \mathcal{L}^n$ is an arbitrary set of positive measure then our assumptions imply $|f|_E|, |g|_E| < \infty$, hence $|f|_E| \cdot |g|_E| = |fg|_E| < \infty$, so fg is bounded on E. But E was an arbitrary set of positive measure, so $\|fg\|_\infty < \infty$. Thus $fg \in L^\infty$. And the inequality holds, since for a.e. x we have
$$|f(x)g(x)| \leqslant \|f\|_\infty |g(x)| \leqslant \|f\|_\infty \|g\|_\infty,$$
so $\|fg\|_\infty \leqslant \|f\|_\infty \|g\|_\infty$.

Now suppose $(p, q, r) \in \overline{\mathbb{R}}^3 \smallsetminus \mathcal{R}$.

- Case 1: $1 \leqslant r \leqslant p, q < \infty$. If $1/r > 1/p + 1/q$, but the desired conclusion fails, since otherwise
$$2^{1/r} = \left\|\left(\frac{2\chi_{B_1(0)}}{\mu(B_1(0))}\right)^2\right\|_r \leqslant \left\|\frac{2\chi_{B_1(0)}}{\mu(B_1(0))}\right\|_p \left\|\frac{2\chi_{B_1(0)}}{\mu(B_1(0))}\right\|_q = 2^{1/p} \cdot 2^{1/q},$$

so $1/r \leqslant 1/p + 1/q$, a contradiction. It fails similarly if $1/r < 1/p + 1/q$, since otherwise

$$\frac{1}{2^r} = \left\| \left(\frac{\chi_{B_1(0)}}{2\mu(B_1(0))} \right)^2 \right\|_r \leqslant \left\| \frac{\chi_{B_1(0)}}{2\mu(B_1(0))} \right\|_p \left\| \frac{\chi_{B_1(0)}}{2\mu(B_1(0))} \right\|_q = \frac{1}{2^p} \cdot \frac{1}{2^q},$$

so $2^{1/p+1/q} \leqslant 2^{1/r}$, and hence $1/r \geqslant 1/p + 1/q$, a contradiction.

- Case 2: $1 \leqslant r \leqslant p < q = \infty$ or $1 \leqslant r \leqslant q < p = \infty$. (Without loss of generality take $1 \leqslant r \leqslant p < q = \infty$.) If $1/p < 1/r$, then the desired conclusion fails, since otherwise

$$\mu(B_1(0))^{1/r} = \|\chi_{B_1(0)}^2\|_r \leqslant \|\chi_{B_1(0)}\|_\infty \|\chi_{B_1(0)}\|_p = 1 \cdot \mu(B_1(0))^{1/p},$$

so $1/r \leqslant 1/p$, a contradiction.

Similarly, if $1/p > 1/r$, then the desired conclusion fails, since otherwise

$$\mu(B_1(0))^{-1/r} = \left\| \left(\frac{\chi_{B_1(0)}}{\mu(B_1(0))} \right)^2 \right\|_r \leqslant \left\| \frac{\chi_{B_1(0)}}{\mu(B_1(0))} \right\|_p \left\| \frac{\chi_{B_1(0)}}{\mu(B_1(0))} \right\|_\infty = \mu(B_1(0))^{-1/p},$$

so $1/p \leqslant 1/r$, a contradiction.

- Case 3: $p = q = r = \infty$. Then the desired conclusion fails, since otherwise

$$\mu(B_1(0)) = \|(\chi_{B_1(0)})^2\|_r \leqslant \|\chi_{B_1(0)}\|_\infty \|\chi_{B_1(0)}\|_\infty = 1 \cdot 1 = 1,$$

which fails for all $n \in \mathbb{Z}_{\geqslant 1}$.

We conclude $\mathcal{R}$ is precisely the set of triples such that the given statement is true. □

4 Homework 3

4.1 Exercise: Folland Exercise 6.21.

If $1 < p < \infty$, $f_n \to f$ weakly in $\ell^p(A)$ if and only if $\sup_n \|f_n\|_p < \infty$ and $f_n \to f$ pointwise.

Solution. Now let $1 < p < \infty$, let $f \in \ell^p(A)$ (we may assume this as mentioned on canvas), and let $q' = p$.

- Suppose $f_n \to f$ weakly in ℓ^p and $q = p'$. Then in particular the ℓ^q function $\chi_{\{a\}}$ has
$$\sum_{a \in A} f_n(a)\chi_{\{a\}} = f_n(a) \to f(a) \text{ as } n \to \infty,$$
so $f_n \to f$ pointwise. For each n, define $\widehat{f}_n(g) = \int g f_n$. Since $f_n \to f$ weakly, the sequence $\{z_n\}_{n=1}^\infty \subset \mathbb{C}$ given by $z_n := \int g f_n$ converges, and hence is bounded in $\mathbb{C}$. Then for all $g \in \ell^q$,
$$\sup_n |\widehat{f}_n(g)| = \sup_n |z_n| < \infty,$$
so
$$\sup_n \|f_n\|_p = \sup_n \|\widehat{f}\| < \infty,$$
where the final inequality is by the uniform boundedness theorem.
- Conversely, suppose that $f_n \to f$ pointwise and $\sup_n \|f_n\|_p < \infty$. Fix $g \in \ell^q = \ell^{p'}$ and $\varepsilon > 0$. We claim $|\langle g, f_n\rangle - \langle g, f\rangle| < \varepsilon$, where $\langle -, *\rangle := \int |(-) \cdot (*)|$. Let $M = \|f\|_p + \sup_n \|f_n\|_p$. Then $M < \infty$ by hypothesis, and we may assume $M > 0$ (since otherwise f_n, and hence f are 0). Since $\|g\|_q^q = \sum_{a\in A} |g(a)|^q < \infty$, we must have $g(a) = 0$ for all but countably many $a \in A$. Thus we may assume $A = \mathbb{Z}_{\geqslant 1}$.

  For all $k \in \{1, \dots, K-1\}$, there exists $N_K \in \mathbb{Z}_{\geqslant 1}$ such that for all $n \geqslant N_k$, $|f_n(k) - f(k)| < \varepsilon/(2(K-1)|g(k)|)$. (If $|g(k)| = 0$, then we may ignore the term $|g(k)||f_n(k) - f(k)| = 0$ in the sum, so this is valid.) Thus, for all $n \geqslant \max\{N_1, \dots, N_k\}$,
$$\sum_{k=1}^{K-1} |g(k)||f_n(k) - f(k)| \leqslant \sum_{k=1}^{K-1} \frac{\varepsilon \cancel{|g(k)|}}{2(K-1)\cancel{|g(k)|}} = \frac{\varepsilon}{2}. \tag{4.1}$$
On the other hand, since $\|g\|_q^q < \infty$, there exists $K \geqslant 2$ such that for all sufficiently large n,
$$\|\chi_{A'} g\|_q^q = \sum_{k=K}^\infty |g(k)|^q < \left(\frac{\varepsilon}{2M}\right)^q.$$
Then, respectively, by Hölder's inequality and the triangle inequality, for all sufficiently large n,
$$\sum_{k=K}^\infty |g(k)||f_n(k) - f(k)| \leqslant \|f_n - f\|_p \|\chi_{A'} g\|_q \leqslant M \frac{\varepsilon}{2M} = \frac{\varepsilon}{2}. \tag{4.2}$$

Thus

$$\begin{aligned} |\langle g, f_n - f\rangle| &= \sum_{k=1}^{\infty} |g(k)||f_n(k) - f(k)| \\ &= \underbrace{\sum_{k=1}^{K-1} |g(k)||f_n(k) - f(k)|}_{<\varepsilon/2 \text{ by } (4.1)} + \underbrace{\sum_{k=K}^{\infty} |g(k)||f_n(k) - f(k)|}_{<\varepsilon/2 \text{ by } (4.2)} < \varepsilon, \end{aligned}$$

so $f_n \to f$ weakly. □

4.2 Exercise: Folland Exercise 6.30.

Let $K\colon (0,\infty) \to [0,\infty)$ such that $\phi(s) \coloneqq \int_0^\infty K(x)x^{s-1}dx < \infty$ for all $0 < s < 1$.

(a) Prove that for $1 < p < \infty$,
$$\iint_{(0,\infty)^2} K(xy)f(x)g(y)dxdy \leqslant \phi\left(\frac{1}{p}\right)\left(\int_0^\infty x^{p-2}f(x)^p dx\right)^{1/p}\left(\int_0^\infty g(x)^q dx\right)^{1/q},$$
where $q = p'$, and $f, g \in L^+((0,\infty))$.

(b) The operator $Tf(x) = \int_0^\infty K(xy)f(y)dy$ is bounded on $L^2((0,\infty))$ with norm $\leqslant \phi(\frac{1}{2})$. (Interesting special case: If $K(x) = e^{-x}$, then T is the Laplace transform and $\phi(s) = \Gamma(s)$.)

Solution.

(a) The integrand of the left-hand side is a nonnegative measurable function (since f, g, and K are), so we can apply Tonelli's theorem below:
$$\begin{aligned}\int_0^\infty\int_0^\infty K(xy)f(x)g(y)\,dx\,dy &= \int_0^\infty\int_0^\infty K(z)\frac{f(z/y)}{y}g(y)\,dz\,dy \ (z \coloneqq xy,\ dx = dz/y)\\ &= \int_0^\infty K(z)\int_0^\infty \frac{f(z/y)}{y}g(y)\,dy\,dz\\ &= \int_0^\infty K(z)\phi_g\left(y\mapsto \frac{f(z/y)}{y}\right)dz\\ &\leqslant \int_0^\infty K(z)\left\|y\mapsto\frac{f(z/y)}{y}\right\|_p \|g\|_q\,dy\,dz\\ &= \int_0^\infty K(z)\left(\int_0^\infty \frac{f(w)^p}{(z/w)^p}\frac{-z}{w^2}\,dw\right)^{1/p}\left(\int_0^\infty g(y)^p\,dy\right)^{1/q}dz\\ &\qquad\text{(substituting } w\coloneqq z/y,\ dy = -z\,dw/w^2)\\ &= \int_0^\infty K(z)z^{-1+1/p}\left(\int_0^\infty f(w)^p w^{p-2}\,dw\right)^{1/p}\left(\int_0^\infty g(y)^p\,dy\right)^{1/q}dz.\end{aligned}$$
Since $\int_0^\infty K(z)z^{1/p-1} = \phi(1/p)$ by definition, the desired inequality follows.

(b) Now consider $p = q = 2$ and define $T\colon L^2((0,\infty)) \to L^2((0,\infty))$ by $f(x) \mapsto \int_0^\infty K(xy)f(y)\,dy$. Then T is linear, and T is bounded since for all $f \in L^2((0,\infty))$,
$$\begin{aligned}\|Tf\|_2^2 &= \int |Tf(y)|^2\,dy\\ &= \int\left|\int K(xy)f(x)\,dx\right|^2 dy\\ &\leqslant \int\left(\int K(xy)|f(x)|\,dx\right)^2 dy \leqslant \phi\left(\frac{1}{2}\right)^2\int x^0|f(x)|^2dx = \phi\left(\frac{1}{2}\right)^2\|f\|_2^2.\end{aligned}$$
where the last inequality is by part (a). Since $f \in L^2((0,\infty))$, this shows $Tf \in$

$L^2((0,\infty))$, so T is indeed a linear map $L^2((0,\infty)) \to L^2((0,\infty))$, and moreover that T is bounded and $\|Tf\|_2 \leqslant \phi\left(\frac{1}{2}\right)\|f\|_2$, which implies $\|T\| \leqslant \phi(1/2)$, as claimed. $\square$

4.3 Exercise: Folland Exercise 6.36.

If $f \in$ weak L^p and $\mu(\{|f| \neq 0\}) < \infty$, then $f \in L^q$ for all $q < p$. On the other hand, if $f \in (\text{weak } L^p) \cap L^\infty$, then $f \in L^q$ for all $q > p$.

Solution. Suppose $f \in$ weak L^p, $0 < q < p$, and $\mu(\{|f| \neq 0\}) < \infty$. Define

$$E_n := \begin{cases} \{0 < |f| \leqslant 1\} & \text{if } n = 0, \\ \{2^{n-1} < |f| \leqslant 2^n\} & \text{if } n \in \mathbb{Z}_{\geqslant 1}. \end{cases}$$

Then $|f| = \sum_{n=0}^\infty \chi_{E_n} |f|$, so

$$\begin{aligned} \|f\|_q^q = \int |f|^q \leqslant \int \Big| \sum_{n=0}^\infty 2^n \chi_{E_n} \Big|^q \leqslant \int \sum_{n=0}^\infty 2^{nq} \chi_{E_n} & \quad \text{(by the triangle inequality)} \\ = \sum_{n=0}^\infty 2^{nq} \mu(E_n) & \quad \text{(by the monotone convergence theorem for series)} \\ = \mu(E_0) + \sum_{n=1}^\infty 2^{nq} \lambda_f(2^{n-1}) & \quad \text{(since } E_n \subset \{|f| > 2^{n-1}\} \text{ and isolating } \mu(E_0)) \\ = \mu(E_0) + \sum_{n=1}^\infty 2^{nq} \lambda_f(2^{n-1}) & \quad \text{(since } [f]_p^p \geqslant 2^{(n-1)p} \lambda_f(2^{n-1}) \text{ by definition of } [f]_p) \\ = \mu(E_0) + \sum_{n=1}^\infty 2^{nq-(np-p)} [f]_p^p, & \\ = \mu(E_0) + \left(\frac{[f]_p}{2} \right)^p \sum_{n=1}^\infty (2^n)^{q-p}, & \end{aligned}$$

which is finite since $E_0 \subset \{|f| \neq 0\}$—which by hypothesis has finite measure—and the infinite sum is a geometric series with ratio $2^{q-p} \in (-1, 1)$ since $q < p$, and thus converges.

Now instead suppose $f \in (\text{weak } L^p) \cap L^\infty$ and $p < q < \infty$. Since f is already L^∞, we can assume $q < \infty$. Define

$$E_n := \begin{cases} \{|f| > 1\} & \text{if } n = 0, \\ \{\frac{1}{2^n} < |f| \leqslant \frac{1}{2^{n-1}}\} & \text{if } n \in \mathbb{Z}_{\geqslant 1}. \end{cases}$$

Computing similarly to before, we have

$$\begin{aligned} \int |f|^q &\leqslant \int \Big(\sum_{n=0}^\infty (2^{1-n})^q \chi_{E_n} \Big) \\ &= \|f\|_\infty^q \mu(E_0) + \sum_{n=1}^\infty 2^{q-nq} \mu(E_n) \\ &\leqslant \|f\|_\infty^q \lambda_f(1) + \sum_{n=1}^\infty 2^{q-nq} \lambda_f(2^{-n}) \\ &\leqslant \|f\|_\infty^q [f]_p^p + \sum_{n=1}^\infty 2^{q-nq+np} [f]_p^p, \end{aligned}$$

which again is finite for the same reasons as before. Thus $f \in L^q$ for all $p < q \leqslant \infty$. □

4.4 Exercise.

The "uncentered" maximal function $\widetilde{M}f$ is defined by $(\widetilde{M}f)(x) = \sup_{x\in B} \frac{1}{m(B)} \int_B |f(y)|dy$ where the supremum is taken over all balls containing x (not only those balls centered at x). Here m denotes Lebesgue measure on $\mathbb{R}^n$.

(a) Obviously $(Mf)(x) \leqslant (\widetilde{M}f)(x)$. Show that there exists a constant c (depending only on the dimension) such that $(\widetilde{M}f)(x) \leqslant c(Mf)(x)$.

(b) Determine explicitly the function $\widetilde{M}(\chi_{[0,1]})$.

(c) It will be shown in class that M and $\widetilde{M}$ are bounded operators on $L^p(\mathbb{R}^n)$ for $1 < p \leqslant \infty$. Does there exist a pair (p, q) with $1 < p, q < \infty$ and $p \neq q$ such that M or $\widetilde{M}$ is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$?

Solution.

(a) Fix $x \in \mathbb{R}^n$, let S be the collection of open balls containing x, let T be the collection of open balls centered at x, and for all Lebesgue measurable subsets E of $\mathbb{R}^n$ define

$$A_E|f| := \frac{1}{m(E)} \int_E |f(y)|\, dy.$$

Since $T \subset S$,

$$Mf(x) = \sup_{E\in T} A_E|f| \leqslant \sup_{E\in S} A_E|f| = \widetilde{M}f(x).$$

For the other inequality, let B_r be any ball containing x of radius r. Then $B \subset B_{2r}(x)$, so

$$\frac{1}{m(B_r)} \int_{B_r} |f(y)|\, dy \leqslant \frac{m(B_{2r}(x))}{m(B_r)} \frac{1}{m(B_{2r}(x))} \int_{B_{2r}(x)} |f(y)|dy \leqslant 2^n Mf(x)$$

Since B was any ball containing x, by taking the supremum over all such balls of all radii we obtain

$$\widetilde{M}f(x) \leqslant 2^n Mf(x).$$

(b) If $B \in S$, then $B = (a, b)$ for some $a, b \in \mathbb{R}$ such that $a < x < b$, so

$$A_B\chi_{[0,1]}(x) = \frac{1}{b-a} \int_{(a,b)} \chi_{[0,1]}(y)\, dy = \begin{cases} 1 & \text{if } (a,b) \subset [0,1], \\ \frac{m((a,b)\cap[0,1])}{b-a} & \text{if } (a,b) \cap [0,1] \neq \varnothing \text{ and } (a,b) \not\subset [0,1], \\ 0 & \text{if } (a,b) \cap [0,1] = \varnothing. \end{cases}$$

We now break into cases:

- If $x \in (0, 1)$ then we can choose a, b such that $0 < a < x < b < 1$, in which case $\widetilde{M}\chi_{[0,1]}(x) = 1$.
- If $x = 0$ (resp. $x = 1$) then by considering the sequence of open intervals $\{E_n = (-1/n, 1)\}_{n=1}^{\infty}$ (resp. $\{E_n = (0, 1 + 1/n)\}_{n=1}^{\infty}$), we see $\widetilde{M}\chi_{[0,1]}(x) = \lim_{n\to\infty} A_{E_n}\chi_{[0,1]}(x) = 1$, so $\widetilde{M}\chi_{[0,1]}(x) = 1$ if $x \in \{0\} \cup \{1\}$.
- If $x < 0$, then for a fixed point $q \in [0, 1]$ and the sequence $\{E_n = (x-1/n, q)\}_{n=1}^{\infty}$,

we have

$$A_{E_n}\chi_{[0,1]}(x) = \frac{m((x-1/n,q)\cap[0,1])}{q-x+1/n} = \frac{q}{q-x+1/n},$$

which tends to $q/(q-x)$ as $n \to \infty$. As a function of $q \in [0,1]$, $q/(q-x)$ is increasing to 1. Thus by taking $q = 1$ and the open sets $\{E_n = (x-1/n, q+1/n)\}_{n=1}^{\infty}$, we conclude that when $x < 0$, $\widetilde{M}\chi_{[0,1]}(x) = \lim_{q\nearrow 1} A_{E_n}\chi_{[0,1]}(x) = \lim_{q\nearrow 1} q/(q-x) = 1/(1-x)$.

– If $x > 1$, then by arguing similarly we obtain $\widetilde{M}\chi_{[0,1]}(x) = 1/x$ if $x > 1$.

We conclude

$$\widetilde{M}\chi_{[0,1]}(x) = \begin{cases} 1 & \text{if } 0 \leqslant x \leqslant 1, \\ 1/(1-x) & \text{if } x < 0, \\ 1/x & \text{if } x > 1. \end{cases}$$ □

(c) No. By part (a) M is bounded if and only if $\widetilde{M}$ is, so it suffices to prove M is not bounded as a map $L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$. Consider an arbitrary $t \in (0,\infty)$ and consider the open cube $(0,t)^n \subset \mathbb{R}^n$. For any $x \in \mathbb{R}^n$, we have

$$\begin{aligned}\|M\chi_{(0,t)^n}\|_q^q &= \int_{\mathbb{R}^n} |M\chi_{(0,t)^n}(x)|^q \,\mathrm{d}x = \int_{\mathbb{R}^n} \left|\sup_{r>0} \frac{1}{m(B_r(x))}\int_{B_r(x)} \chi_{(0,t)^n}(y)\,\mathrm{d}y\right|^q \mathrm{d}x \\ &= \int_{\mathbb{R}^n} \left|\sup_{r>0} \frac{m(B_r(x)\cap(0,t)^n)}{m(B_r(x))}\right|^q \mathrm{d}x = \int \chi_{(0,t)^n}(x)\,\mathrm{d}x = m((0,t)^n),\end{aligned}$$

so $\|M\chi_{(0,t)^n}\|_q = m((0,t)^n)^{1/q} = t^{n/q}$. On the other hand, for an arbitrary constant C,

$$C\|\chi_{(0,t)^n}\|_p = Cm((0,t)^n)^{1/p} = Ct^{n/p}.$$

If M were bounded as an operator $L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$, then there exists a constant C such that for all $t \in (0,\infty)$, $t^{n/q} \leqslant Ct^{n/p}$, or equivalently, such that

$$t^{n(\frac{1}{q}-\frac{1}{p})} \leqslant C.$$

But this cannot be true at all $t \in (0,\infty)$ since p, q, n are fixed; by choosing sufficiently small t when $1/p > 1/q$ or sufficiently large t (when $1/p < 1/q$), this fails. Thus M, hence also $\widetilde{M}$, is unbounded as an operator $L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$.

5 Homework 4

5.1 Exercise: Folland Exercise 6.41.

Suppose $1 < p \leqslant \infty$ and $p^{-1} + q^{-1} = 1$. If T is a bounded linear operator on L^p such that $\int (Tf)g = \int f(Tg)$ for all $f, g \in L^p \cap L^q$, then T extends uniquely to a bounded operator on L^r for all r in $[p, q]$ (if $p < q$) or $[q, p]$ (if $q < p$). If $p = \infty$, further assume that μ is semifinite.

Solution. Let $p \in (1, \infty]$, let $q = (p-1)/p$, let Σ be the set of simple functions that vanish outside a set of finite measure, and let r lie in the closed interval between p and q.

5.2 Claim. T maps $L^p \cap L^q$ into L^q and is bounded as a map $L^p \cap L^q \to L^q$.

Proof. Let $f \in L^p \cap L^q$. Then $Tf \in L^p$ by hypothesis. Thus if $p < \infty$ then $|Tf|^p \in L^1$ (since $Tf \in L^p$), so $\{|Tf|^p \neq 0\} = \{Tf \neq 0\}$ is σ-finite by Folland Proposition 2.23(a). On the other hand, if $p = \infty$ then μ is semifinite by hypothesis. In either case, it follows from Folland Theorem 6.14 that

$$\|Tf\|_q = \sup\left\{ \left| \int g(Tf) \right| \;\middle|\; g \in \Sigma \text{ and } \|g\|_p = 1 \right\}, \tag{5.1}$$

so it suffices to show the right-hand side is finite. To that end, suppose $g \in \Sigma$ and $\|g\|_p = 1$. We have $g \in L^q$ since $g \in \Sigma$, so in particular $g \in L^p \cap L^q$. Then

$$\begin{aligned} \left| \int g(Tf) \right| &= \left| \int f(Tg) \right| && \text{(by our hypothesis on } T\text{)} \\ &\leqslant \|f\|_p \|Tg\|_q && \text{(by Hölder's inequality)} \\ &\leqslant \|f\|_p \|T\|_{L^p \to L^p} \|g\|_p \\ &\leqslant \|f\|_p \|T\|_{L^p \to L^p} && \text{(since } \|g\|_p = 1\text{).} \end{aligned}$$

Our above estimate is independent of our choice of g, so by Equation (5.1)

$$\|Tf\|_q \leqslant \|T\|_{L^p \to L^p} \|f\|_p.$$

Thus T maps $L^p \cap L^q$ into L^q and is bounded as a map $(L^p \cap L^q, \|-\|_p) \to (L^q, \|-\|_q)$. □

5.3 Claim. The map

$$\begin{aligned} \widetilde{T} \colon L^p + L^q &\longrightarrow L^p + L^q, \\ f + g = h &\longmapsto \widetilde{T}g := Tf + \lim_{n\to\infty} Tg_n, \end{aligned}$$

where $\{g_n\}_{n=1}^\infty \subset L^p \cap L^q$ and $g_n \to g$ in L^q, is a well-defined bounded linear operator.

Proof.

- $\widetilde{T}$ *is well-defined*: Let $g \in L^p + L^q$. Since $L^p \cap L^q$ is dense in $L^p + L^q$ (because $L^p \cap L^q$ contains Σ, which is a dense subset in both L^p and L^q), such an approximating sequence $\{g_n\}_{n=1}^\infty$ as in the claim exists in $L^p \cap L^q$.

Next we show $\widetilde{T}$ is independent of the choice of sequence $\{g_n\}_{n=1}^\infty \subset L^p \cap L^q$. Since $\{g_n\}_{n=1}^\infty$ is Cauchy in L^q and T is bounded as a map $L^p \cap L^q \to L^q$ by the first claim,

$$\|Tg_n - Tg_m\|_q = \|T(g_n - g_m)\|_q \leqslant \|T\|_{L^p \to L^q} \|g_n - g_m\|_q \to 0$$

as $n, m \to \infty$. By uniqueness of the limit (as L^q is a Banach space), we conclude $\widetilde{T}g$ is independent of the choice of approximating sequence sequence.

- *$\widetilde{T}$ is linear*: We are given $\widetilde{T}$ is linear on L^q, so it suffices to show linearity on L^p. Suppose $g, g' \in L^p \cap L^q$, $\alpha \in \mathbb{C}$, $\{g_n\}_{n=1}^\infty, \{g'_n\}_{n=1}^\infty \subset L^p \cap L^q$, and $g_n \to g, g'_n \to g'$ in L^q. Then

$$\begin{aligned}
\widetilde{T}(\alpha g + g') &= \lim_{n\to\infty} T(\alpha g + g') \\
&= \lim_{n\to\infty} (\alpha T g_n + T g'_n) && \text{(by linearity of } T\text{)} \\
&= \alpha \lim_{n\to\infty} T g_n + \lim_{n\to\infty} T g'_n && \text{(by linearity of limits that exist)} \\
&= \alpha \widetilde{T} g + \widetilde{T} g' && \text{(by definition of } \widetilde{T}\text{)}.
\end{aligned}$$

Hence $\widetilde{T}$ is linear.

- *$\widetilde{T}$ is bounded as a map $L^q \to L^q$*: Let $g \in L^q$ and let $\{g_n\}_{n=1}^\infty \subset L^p \cap L^q$ such that $g_n \to g$ in L^q. Since $q < \infty$ by hypothesis, we can write

$$\begin{aligned}
\|\widetilde{T}g\|_q^q = \int |\widetilde{T}g|^q &= \int \Big|\lim_{n\to\infty} T g_n\Big|^q \\
&= \int \lim_{n\to\infty} |T g_n|^q && \text{(by continuity of } \mathbb{R} \ni x \mapsto |x|^q \in \mathbb{R}\text{)} \\
&\leqslant \liminf_{n\to\infty} \|T g_n\|_q^q && \text{(by Fatou's lemma)} \\
&\leqslant \|T\|_{L^p\to L^q}^q \liminf_{n\to\infty} \|g_n\|_q^q \\
& && \text{(since } T \text{ is bounded as an operator } L^p \cap L^q \to L^q\text{)} \\
&= \|T\|_{L^p\to L^q}^q \lim_{n\to\infty} \|g_n\|_q^q \\
& && \text{(since } \lim_{n\to\infty}\|g_n\|_q^q \text{ exists, hence equals the liminf; see below)} \\
&= \|T\|_{L^p\to L^q}^q \|g\|_q^q.
\end{aligned}$$

The penultimate equality here follows from the fact $g_n \to g$ in L^q, since for all $\varepsilon > 0$ and all sufficiently large n,

$$\|g_n\|_q \leqslant \|g\|_q + \|g_n - g\|_q < \|Tg\|_q + \varepsilon < \infty;$$

taking the qth power, we obtain $\|g_n\|_q^q \leqslant (\|g\|_q + \varepsilon)^q < \infty$, so $\lim_{n\to\infty}\|g_n\|_q^q = \|g\|_q^q$. ☐

5.4 Claim. $\widetilde{T}$ is the unique bounded operator on L^r for all r in the interval $[p, q]$ (if $p < q$) or $[q, p]$ (if $q < p$) that extends T.

Proof. Since $\widetilde{T}$ is strong type (p, p) and strong type (q, q), by the Riesz–Thorin theorem $\widetilde{T}$ is strong-type (r, r) for all r in the interval $[p, q]$ (if $p < q$) or $[q, p]$ (if $q < p$). To see

$\widetilde{T}$ is the unique such extension, suppose S is another such extension of T. We can write each $h \in L^r$ as a sum $h = f + g$ for some $f \in L^p$ and $g \in L^r$, so

$$Sh = S(f + g) = Sf + Sg = \widetilde{T}f + \widetilde{T}g = \widetilde{T}h$$

since because S is an extension we have $Sf = \widetilde{T}f$ for all $f \in L^p$ and $Sg = \widetilde{T}g$ for all $g \in L^q$. Thus $S = \widetilde{T}$, so the extension is unique. □

5.5 Exercise: Folland Exercise 6.42.

Prove the Marcinkiewicz theorem in the case $p_0 = p_1$. (Setting $p = p_0 = p_1$, we have $\lambda_{Tf}(\alpha) \leqslant (C_0\|f\|_p/\alpha)^{q_0}$ and $\lambda_{Tf}(\alpha) \leqslant (C_1\|f\|_p/\alpha)^{q_1}$. Use whichever estimate is better, depending on α, to majorize $q\int_0^\infty \alpha^{q-1}\lambda_{Tf}(\alpha)d\alpha$.)

Proof. Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are measure spaces; $p, q_0, q_1 \in [1, \infty]$ and $p \leqslant q_0, q_1$, and $q_0 \neq q_1$; and

$$\frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}, \qquad \text{where } 0 < t < 1.$$

Let $T\colon L^p(\mu) \to L^0(\nu)$ be[2] a sublinear map of weak types (p, q_0) and (p, q_1). We claim T is strong type (p, q). More precisely, suppose $[Tf]_{q_j} \leqslant C_j\|f\|_p$ for $j = 0, 1$. We claim $\|Tf\|_q \leqslant B_p\|f\|_p$ where B_p depends only on p, q_j, and C_j in addition to p.

Then for $\alpha > 0$ we have the estimates

$$\lambda_{Tf}(\alpha) \leqslant (C_0\|f\|_p/\alpha)^{q_0} \qquad \text{and} \qquad \lambda_{Tf}(\alpha) \leqslant (C_1\|f\|_p/\alpha)^{q_1},$$

so we obtain the estimate

$$\begin{aligned}
\|Tf\|_q^q = \int |Tf|^q &= q\int_0^\infty \alpha^{q-1}\mu\{|Tf| > \alpha\}\,\mathrm{d}\alpha \\
&= q\int_0^{\|f\|_p} \alpha^{q-1}\mu\{|Tf| > \alpha\}\,\mathrm{d}\alpha + q\int_{\|f\|_p}^\infty \alpha^{q-1}\mu\{|Tf| > \alpha\}\,\mathrm{d}\alpha \\
&\leqslant q\int_0^{\|f\|_p} \alpha^{q-1}\left(\frac{C_0\|f\|_p}{\alpha}\right)^{q_0}\mathrm{d}\alpha + q\int_{\|f\|_p}^\infty \alpha^{q-1}\left(\frac{C_1\|f\|_p}{\alpha}\right)^{q_1}\mathrm{d}\alpha \\
&\leqslant qC_0^{q_0}\|f\|_p^{q_0}\int_0^{\|f\|_p} \alpha^{q-q_0-1}\,\mathrm{d}\alpha + qC_1^{q_1}\|f\|_p^{q_1}\int_{\|f\|_p}^\infty \alpha^{q-q_1-1}\,\mathrm{d}\alpha \\
&\leqslant qC_0^{q_0}\|f\|_p^{q_0}\left[\frac{\alpha^{q-q_0}}{q-q_0}\right]_{\alpha=0}^{\alpha=\|f\|_p} + qC_1^{q_1}\|f\|_p^{q_1}\left[\frac{\alpha^{q-q_1}}{q-q_1}\right]_{\alpha=\|f\|_p}^{\alpha=\infty}. \\
&= \left(\frac{qC_0^{q_0}\|f\|_p^{q_0}\|f\|^{q-q_0}}{q-q_0}\right) - \left(\frac{qC_1^{q_1}\|f\|_p^{q_1}\|f\|_p^{q-q_1}}{q-q_1}\right) \\
&= \left(\frac{qC_0^{q_0}}{q-q_0} + \frac{qC_1^{q_1}}{q_1-q}\right)\|f\|_p^q.
\end{aligned}$$

Thus T is strong type (p, q), as claimed, and moreover $B_p \coloneqq \left(\frac{qC_0^{q_0}}{q-q_0} + \frac{qC_1^{q_1}}{q_1-q}\right)^{1/q}$ depends only on q_j and C_j for $j = 0, 1$. □

5.6 Exercise: Folland Exercise 6.45, Altered.

The following concerns Folland Exercise 6.45, which reads as follows:

> If $0 < \alpha < n$, define an operator T_α on functions on $\mathbb{R}^n$ by
> $$T_\alpha f(x) := \int |x-y|^{-\alpha} f(y)\,\mathrm{d}y$$
> Then T_α is weak type $(1,(n-\alpha)^{-1})$ and strong type (p,r) with respect to Lebesgue measure on $\mathbb{R}^n$, where $1 < p < n\alpha^{-1}$ and $r^{-1} = p^{-1} - \alpha n^{-1}$. (The case $n = 3$, $\alpha = 1$ is of particular interest in physics: If f represents the density of a mass or charge distribution, $-(4\pi)^{-1}T_1 f$ represents the induced gravitational or electrostatic potential.)

The following aims to correct this exercise.

(a) Use a scaling argument to show that the exercise is incorrect as stated.

(b) Replace the exponent $-\alpha$ in the definition of with $-n+\alpha$ in the question. Prove that (this version of) T_α is weak type $(1, 1(n-\alpha)^{-1})$ and strong type (p,r) under the conditions on α, p, and r as stated in the exercise. Hint: First show that T_α is of *weak* type (p,r).

Solution.

(a) Suppose for a contradiction T_α is strong type (p,r), so that $\|T_\alpha\|_{L^p\to L^r} < \infty$. Now fix $\varepsilon > 0$. Since $\|T_\alpha\|_{L^p\to L^r} = \sup\{\|T_\alpha f\|_r \mid \|f\|_p = 1\} < \infty$, there exists $f \in L^p$ such that $\|f\|_p = 1$ and
$$\|T_\alpha f\|_r > (1-\varepsilon)\|T_\alpha\|_{L^p\to L^r} \tag{5.2}$$
For each $b \in \mathbb{R}_{>0}$ define $g_b \colon \mathbb{R}^n \to \mathbb{C}$ by
$$g_b(x) = f(bx).$$
Then $g_b(x) \in L^p$ and for a fixed $b \in \mathbb{R}_{>0}$ be fixed. We have
$$\|g_b\|_p^p = \int |g_b(x)|^p\,\mathrm{d}x = \frac{1}{b^{np}}\int |f(x)|^p\,\mathrm{d}x = \frac{1}{b^{np}},$$
so $\|g_b\|_p = 1/b^n$. And for each $x \in \mathbb{R}^n$, we have
$$\begin{aligned} T_\alpha g_b(x) &= \int |x-y|^{-\alpha} f(by)\,\mathrm{d}y \\ &= b^{-n}\int |x-y/b|^{-\alpha} f(y)\,\mathrm{d}y && \text{(substitute } by \mapsto y\text{)} \\ &= b^{-n}\int \left|\frac{bx-y}{b}\right|^{-\alpha} f(y)\,\mathrm{d}y = b^{\alpha-n}\int |bx-y|^{-\alpha} f(y)\,\mathrm{d}y, \end{aligned}$$

so

$$\begin{aligned}\|T_\alpha g_b\|_r^r &= b^{r(\alpha-n)} \int \left| \int |bx-y|^{-\alpha} f(y)\,\mathrm{d}y \right|^r \mathrm{d}x \\ &= b^{r(\alpha-n)} \int \left| b^{-n} \int |x-y|^{-\alpha} f(y)\,\mathrm{d}y \right|^r \mathrm{d}x \qquad \text{(substitute } bx \mapsto x\text{)} \\ &= b^{r(\alpha-2n)} \int |T_\alpha f(x)|^r \,\mathrm{d}x = b^{r(\alpha-2n)} \|T_\alpha f\|_r^r.\end{aligned}$$

Thus

$$b^{\alpha-n}\|T_\alpha f\|_r = \frac{b^{\alpha-2n}\|T_\alpha f\|_r}{b^{-n}} = \frac{\|T_\alpha g_b\|_r}{\|g_b\|_p} \leqslant \|T_\alpha\|_{L^p \to L^r}.$$

Therefore, since $0 < \alpha < n$ and in particular $\alpha \neq n$, we can choose $f \in L^p$ and $b > 0$ sufficiently large such that the left-hand side is strictly larger than the right-hand side (since otherwise T_α is the zero operator, contrary to the given definition of T_α), which contradicts the assumed boundedness of T on L^p. It follows that Folland Exercise 6.45 is incorrect as stated.

(b) Define $K\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ by

$$K(x,y) \coloneqq |x-y|^{-\alpha}.$$

Then K is $m \times m$-measurable, and for each $x \in \mathbb{R}^n$ and $\beta > 0$ we have

$$\begin{aligned}\lambda_{K(x,-)}(\beta) &= m(\{y \in \mathbb{R}^n \mid |x-y|^{-\alpha} > \beta\}) \\ &= m(\{y \in \mathbb{R}^n \mid |x-y| < \beta^{-1/\alpha}\}) \\ &\leqslant m(B_{\beta^{-1/\alpha}}(x))\end{aligned}$$

Since the measure of a ball of radius r in $\mathbb{R}^n$ is a scalar multiple of the radius to the power of n, there exists $C > 0$ such that for all $x \in \mathbb{R}^n$ and all $\beta > 0$,

$$m(B_{\beta^{-1/\alpha}}(x)) = C\,\beta^{-n/\alpha}$$

and thus

$$\beta^{n/\alpha}\lambda_{K(x,-)}(\beta) \leqslant \beta^{n/\alpha} m(B_{\beta^{-1/\alpha}}(x)) = \beta^{-n/\alpha}\beta^{n/\alpha}C = C.$$

Thus, by taking the $1/(n/a)$th power of both sides and taking the supremum over all $\beta \in \mathbb{R}_{>0}$, we obtain for all $x \in \mathbb{R}^n$ that

$$[K(x,-)]_q = \sup_{q>0}(\beta^q \lambda_{K(x,-)}(\beta))^{1/q} \leqslant C^{1/q}.$$

Arguing identically (but replacing $K(x,-)$ with $K(-,y)$ and x with y), there exists $C' > 0$ such that $[K(-,y)]_q \leqslant C'^{1/q}$ for all $y \in \mathbb{R}^n$. Now replacing C with the maximum of $C^{1/q}, C'^{1/q}$, the result then follows immediately from Folland Theorem 6.36. □

5.7 Exercise: Folland Exercise 8.4.

If $f \in L^\infty$ and $\|\tau_y f - f\|_\infty \to 0$ as $y \to 0$, then f agrees a.e. with a uniformly continuous function. (Let $A_r f$ be as in Folland Theorem 3.18. Then $A_r f$ is uniformly continuous for $r > 0$ and uniformly Cauchy as $r \to 0$.)

Solution. The statement of Exercise 5.7 follows immediately from the following points:

(i) $A_{1/n}f(x) \to f(x)$ a.e. as $n \to \infty$.
(ii) For all $n \in \mathbb{Z}_{\geqslant 1}$, $A_{1/n}f(x)$ is uniformly continuous as a function of $x \in \mathbb{R}^n$.
(iii) The sequence $\{A_{1/n}f\}_{n=1}^\infty$ is uniformly Cauchy.
(iv) If $\{f_n : \mathbb{R}^n \to \mathbb{C}\}_{n=1}^\infty$ is a uniformly Cauchy sequence of uniformly continuous functions, then $\lim_{n\to\infty} f_n$ is uniformly continuous.

Proof of (i). This is just Folland Theorem 3.18 since L^∞ functions are L^1_{loc}. □

Proof of (ii). Let $n \in \mathbb{Z}_{\geqslant 1}$. Fix $\varepsilon > 0$. It suffices to show $\|\tau_y A_r f - A_r f\|_u \to 0$ as $y \to 0$. For any x, we have

$$\begin{aligned}
|\tau_y A_{1/n}f(x) - A_{1/n}f(x)| &= \frac{1}{m(B_r(0))}\left|\int_{B_r(x-y)} |f(z)|\,\mathrm{d}z - \int_{B_r(x)} |f(z)|\,\mathrm{d}z\right| \\
&= \frac{1}{m(B_r(0))}\left|\int_{B_r(x-y)} |f(z)|\,\mathrm{d}z - \int_{B_r(x-y)} |f(z-y)|\,\mathrm{d}z\right| \\
&\qquad \text{(substitute } z \mapsto z-y\text{)} \\
&\leqslant \frac{1}{m(B_r(0))}\int_{B_r(x-y)} |\tau_y f(z) - f(z)|\,\mathrm{d}z \\
&\leqslant \frac{1}{m(B_r(0))}\int_{B_r(x-y)} \|\tau_y f - f\|_\infty\,\mathrm{d}z \\
&\qquad \text{(since } |\tau_y f(z) - f(z)| \leqslant \|\tau_y f - f\|_\infty \text{ for a.e. } z \in \mathbb{R}^n\text{)} \\
&= \|\tau_y f - f\|_\infty.
\end{aligned}$$

Taking the supremum of both sides over all $x \in \mathbb{R}^n$, we obtain

$$\|\tau_y A_{1/n}f - A_{1/n}f\|_u \leqslant \|\tau_y f - f\|_\infty.$$

Since $\|\tau_y f - f\|_\infty \to 0$ as $y \to 0$ by hypothesis, we conclude $A_{1/n}f$ is uniformly continuous. □

Proof of (iii). We claim $\|A_{1/n}f - A_{1/m}f\|_u \to 0$ as $m, n \to \infty$. Fix $\varepsilon > 0$. Since $A_{1/n}$ By Folland Lemma 3.16, $A_r f$ is a continuous function of $r \in \mathbb{R}_{>0}$; thus $A_{1/n}f - A_{1/m}f$ is continuous for all $n, m \in \mathbb{Z}_{\geqslant 1}$, so its supremum norm equals its infinity norm. Hence

$$\|A_{1/n}f - A_{1/m}f\|_u = \|A_{1/n}f - A_{1/m}f\|_\infty \leqslant \|A_{1/n}f - f\|_\infty + \|A_{1/m}f - f\|_\infty. \tag{5.3}$$

Where $n \in \mathbb{Z}_{\geqslant 1}$, we have

$$\|A_{1/n}f - f\|_\infty = \left\| x \mapsto \frac{1}{m(B_{1/n}(x))}\int_{B_{1/n}(x)} |f(y)|\,\mathrm{d}y - f(x)\right\|_\infty$$

$$\begin{aligned}
&\leqslant \left\| x \mapsto \frac{1}{m(B_{1/n}(x))} \int_{B_{1/n}(x)} |f(y) - f(x)| \,\mathrm{d}y \right\|_\infty \\
&\hfill \text{(by the triangle inequality)} \\
&\leqslant \left\| x \mapsto \frac{1}{m(B_{1/n}(x))} \int_{B_{1/n}(0)} |\tau_y f(x) - f(x)| \,\mathrm{d}y \right\|_\infty \\
&\leqslant \frac{1}{m(B_{1/n}(x))} \int_{B_{1/n}(0)} \| x \mapsto |\tau_y f(x) - f(x)| \|_\infty \,\mathrm{d}y \\
&\leqslant \frac{1}{m(B_{1/n}(x))} \int_{B_{1/n}(0)} \|\tau_y f - f\|_\infty \,\mathrm{d}y, \longrightarrow 0 \text{ as } n \to \infty
\end{aligned}$$

where we used Minkowski's inequality for integrals (Folland Theorem 6.19) since $\tau_y f - f \in L^\infty$ for a.e. $y \in \mathbb{R}^n$ and $[y \mapsto \|\tau_y f - f\|_p] \in L^1$.

Thus both terms on the right-hand side of Equation (5.3) tend to 0 as $m, n \to \infty$, so $\{A_{1/n} f\}_{n=1}^\infty$ is uniformly Cauchy. □

Proof of (iv). Fix $\varepsilon > 0$ and $g = \lim_{n\to\infty} f_n$. Then for all sufficiently large n, $|f_n(x) - g(x)| < \varepsilon/3$. Since each f_n is uniformly continuous, there exists $\delta > 0$ such that $|f_n(x) - f_n(y)| < \varepsilon/3$ whenever $|x - y| < \delta$. Thus, for any x, y such that $|x - y| < \delta$ and all sufficiently large n, we have

$$\begin{aligned}
|g(x) - g(y)| &\leqslant |g(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - g(y)| \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,
\end{aligned}$$

so g is uniformly continuous. □

6 Homework 5

Folland Exercise 8.13.

Let $f(x) = \frac{1}{2} - x$ on the interval $[0, 1)$, and extend f to be periodic on $\mathbb{R}$.
(a) $\widehat{f}(0) = 0$, and $\widehat{f}(\kappa) = (2\pi i\kappa)^{-1}$ if $\kappa \neq 0$.
(b) $\sum_1^\infty k^{-2} = \pi^2/6$. (Use Parseval's identity.)

Solution.

(a) First note $f \in L^2(\mathbb{T})$, since

$$\|f\|_2^2 = \int_{\mathbb{T}} |f(x)|^2 \,\mathrm{d}x = \left[\frac{x}{4} - \frac{x^2}{2} + \frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{1}{12}. \tag{6.1}$$

We have

$$\widehat{f}(0) = \int_{\mathbb{T}} f(x)e^{-2\pi i 0\cdot x} \,\mathrm{d}x = \int_0^1 f(x) = \frac{1}{2} - \left[\frac{x^2}{2}\right]_{x=0}^{x=1} = 0,$$

and if $k \in \mathbb{Z} \smallsetminus \{0\}$,

$$\begin{aligned}\widehat{f}(k) &= \int_{\mathbb{T}} \left(\frac{1}{2} - x\right) e^{-2\pi ikx} \,\mathrm{d}x = \int_0^1 \frac{1}{2} e^{-2\pi ikx} \,\mathrm{d}x - \int_0^1 xe^{-2\pi ikx} \,\mathrm{d}x \\ &= \frac{-1}{4\pi ik} - \left[\frac{x}{-2\pi ike^{-2\pi ikx}}\right]_{x=0}^{x=1} + \frac{-1}{2\pi ik} \int_0^1 e^{-2\pi ikx} \,\mathrm{d}x \\ &= \frac{-1}{4\pi ik} - \frac{-1}{2\pi ik} + \frac{1}{4\pi ik} = \frac{1}{2\pi ik}.\end{aligned}$$

(b) By part (a) $|\widehat{f}(k)|^2 = 1/(4\pi^2k^2)$, so by Plancherel's theorem

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 4\pi^2 \sum_{k=0}^{\infty} |\widehat{f}(k)|^2 = 2\pi^2 \sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^2 = 2\pi^2 \|f\|_2^2 \overset{(6.1)}{=} \frac{\pi^2}{6}. \qquad \square$$

Folland Exercise 8.15.

Fix $a > 0$. Define $\operatorname{sinc} 0 = 1$ and $\operatorname{sinc} x = (\sin \pi x)/\pi x$ for $x \in \mathbb{R} \setminus \{0\}$.

(a) $\widehat{\chi}_{[-a,a]}(x) = \chi^{\vee}_{[-a,a]}(x) = 2a \operatorname{sinc} 2ax$.

(b) Let

$$\mathcal{H}_a := \{f \in L^2 \mid \widehat{f}(\xi) = 0 \text{ a.e. whenever } |\xi| > a\}.$$

Then $\mathcal{H}_a$ is a Hilbert space and $\left\{\sqrt{2a} \operatorname{sinc}(2ax - k) \mid k \in \mathbb{Z}\right\}$ is an orthonormal basis for $\mathcal{H}_a$.

(c) (The sampling theorem). If $f \in \mathcal{H}_a$, then $f \in C_0$ (after modification on a null set), and $f(x) = \sum_{-\infty}^{\infty} f(k/2a) \operatorname{sinc}(2ax - k)$, where the series converges both uniformly and in L^2.[3]

Solution.

(a) We have

$$\widehat{\chi}_{[-a,a]}(\xi) = \int_{-a}^{a} e^{-2\pi i \xi x}\, \mathrm{d}x = \frac{-1}{2\pi i \xi}(e^{-2\pi i \xi a} - e^{2\pi i \xi a}) = \frac{\sin(2\pi a \xi)}{\pi \xi} = 2a \operatorname{sinc}(2a\xi)$$

and, by changing variables $x \mapsto -x$ in the integrand of $\chi^{\vee}_{[-a,a]}(\xi)$, we find

$$\chi^{\vee}_{[-a,a]}(\xi) = \int_{-a}^{a} e^{2\pi i \xi x} = -\int_{a}^{-a} e^{-2\pi i \xi x}\, \mathrm{d}x = \int_{-a}^{a} e^{-2\pi i \xi x}\, \mathrm{d}x = \chi^{\wedge}_{[-a,a]}(\xi).$$

(b) $\mathcal{H}_a$ is a linear subspace: If $f, g \in \mathcal{H}_a$ and $\lambda \in \mathbb{C}$, then for all $|\xi| > a$ we have $\widehat{f}(\xi) = \widehat{g}(\xi) = 0$, so

$$(f + \lambda g)^{\wedge}(\xi) = \widehat{f}(\xi) + \lambda \widehat{g}(\xi) = 0 + \lambda 0 = 0.$$

Thus $f + \lambda g \in \mathcal{H}_a$, so $\mathcal{H}_a$ is a linear subspace of L^2.

$\mathcal{H}_a$ is closed: Suppose $\{f_n\}_{n=1}^{\infty} \subset \mathcal{H}_a$ and $\|f_n - f\|_2 \to 0$. Since the Fourier transform is unitary on L^2 (hence an isometry), $\|\widehat{f}_n - \widehat{f}\|_2 \to 0$, that is, $f_n \to f$ in L^2. Thus there exists a subsequence $\widehat{f}_{n_k} \to \widehat{f}$ pointwise a.e., so for a.e. $x \in \mathbb{R}$, we have for $|\xi| > a$

$$\widehat{f}(\xi) = \lim_{k\to\infty} \widehat{f}_{n_k}(x) = \lim_{k\to\infty} 0 = 0.$$

Thus $f \in \mathcal{H}_a$, so $\mathcal{H}_a$ is a closed linear subspace of the Hilbert space L^2, and thus $\mathcal{H}_a$ is a Hilbert space.

Now for $k \in \mathbb{Z}$ and $x \in \mathbb{R}$, define $\zeta_k(x) := \sqrt{2a} \operatorname{sinc}(2ax - k)$. We claim $\{\zeta_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $\mathcal{H}_a$. We first show $\{\zeta_k\}_{k\in\mathbb{Z}} \subset \mathcal{H}_a$. For any $k \in \mathbb{Z}$,

$$\zeta_k(x) = \sqrt{2a} \operatorname{sinc}(2ax - k) = \frac{1}{\sqrt{2a}}(2a \operatorname{sinc}(2a(x - k/2a))) \overset{(a)}{=} \frac{1}{\sqrt{2a}} \chi^{\vee}_{[-a,a]}(x - k/2a). \tag{6.2}$$

Taking the Fourier transform, we obtain

$$\zeta_k^\wedge(\xi) = \frac{1}{\sqrt{2a}}(\tau_{k/2a}\chi_{[-a,a]}^\vee)^\wedge(\xi) = \frac{e^{-2\pi i\xi(k/2a)}}{\sqrt{2a}}(\chi_{[-a,a]}^\vee)^\wedge(\xi) = \frac{e^{-2\pi i(k/2a)\xi}}{\sqrt{2a}}\chi_{[-a,a]}(\xi), \tag{6.3}$$

where for the last equality we used $\chi_{[-a,a]} \in L^2$ and that the Fourier transform is a unitary isomorphism on L^2. In particular, Equation (6.3) shows both that $\zeta_k \in L^2$ (since its Fourier transform is) and that $\widehat{\zeta}_k(\xi) = 0$ whenever $|\xi| > a$, so $\zeta_k \in \mathscr{H}_a$.

$\{\zeta_k\}_{k\in\mathbb{Z}}$ is an orthonormal set in $\mathscr{H}_a$: Since the Fourier transform is a unitary operator $L^2 \to L^2$, we have for all $k \in \mathbb{Z}$ that

$$\langle\zeta_k|\zeta_k\rangle = \langle\zeta_k^\wedge|\zeta_k^\wedge\rangle = \frac{1}{2a}\int_{-a}^a e^{2\pi i(k-k)\xi}\,\mathrm{d}\xi = \frac{1}{2a}\int_{-a}^a 1\,\mathrm{d}\xi = 1,$$

and if $\ell \in \mathbb{Z} \setminus \{k\}$,

$$\begin{aligned}\langle\zeta_k|\zeta_\ell\rangle = \langle\zeta_k^\wedge|\zeta_\ell^\wedge\rangle &\overset{(6.3)}{=} \frac{1}{2a}\int e^{-2\pi i(k/2a)\xi}\chi_{[-a,a]}(\xi)\overline{e^{-2\pi i\xi(\ell/2a)}\chi_{[-a,a]}(\xi)}\,\mathrm{d}\xi \\ &= \frac{1}{2a}\int_{-a}^a e^{2\pi i\xi\left(\frac{k-\ell}{2a}\right)}\,\mathrm{d}\xi = \frac{1}{2a}\left(\frac{2a}{2\pi i(k-\ell)}(e^{\pi i(k-\ell)} - e^{\pi i(\ell-k)})\right) = \frac{\sin(\pi(k-\ell))}{2\pi i(k-\ell)} = 0.\end{aligned}$$

Thus $\{\zeta_k\}_{k\in\mathbb{Z}}$ is an orthonormal set in $\mathscr{H}_a$.

$\{\zeta_k\}_{k\in\mathbb{Z}}$ is a basis of $\mathscr{H}_a$: Suppose $f \in \mathscr{H}_a$ satisfies $\langle f|\zeta_k\rangle = 0$ (and hence also $\langle f^\wedge|\zeta_k^\wedge\rangle = 0$) for all $k \in \mathbb{Z}$. Then for each $k \in \mathbb{Z}$,

$$\begin{aligned}0 = \int f^\wedge(\xi)\overline{\zeta_k^\wedge(\xi)}\,\mathrm{d}\xi &\overset{(6.3)}{=} \frac{1}{\sqrt{2a}}\int_{-a}^a f^\wedge(\xi)e^{2\pi i(k/2a)\xi}\,\mathrm{d}\xi \\ &= \frac{1}{\sqrt{2a}}\int_{-1/2}^{1/2} f^\wedge(\eta/2a)e^{2\pi ik\eta}\,\mathrm{d}\eta = \sqrt{2a}\int_{\mathbb{T}} f^\wedge(-\eta/2a)\overline{E_k}(\eta)\,\mathrm{d}\eta = \sqrt{2a}\langle\widehat{f}\circ s|E_k\rangle,\end{aligned}$$

where $s\colon \eta \mapsto -\eta/2a$, and $E_k(\eta) = e^{2\pi ik\eta}$. In particular $\langle\widehat{f}\chi_{[-a,a]}|E_k\rangle = 0$ for all $k \in \mathbb{Z}$. But by Folland Theorem 8.20 $\{E_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{T})$, so $\widehat{f}\chi_{[-a,a]} = 0$ a.e. Therefore, since $\widehat{f} \in L^2(\mathbb{T})$, by the Fourier inversion theorem (namely since the Fourier transform is an isomorphism $L^2 \to L^2$), $\widehat{f}\circ s = 0$ a.e. on $[-1/2, 1/2]$. Thus $\widehat{f}\chi_{[-a,a]} = 0$ a.e., and hence $\widehat{f} = 0$ for a.e. $\xi \in \mathbb{R}$ (since already $\widehat{f}(\xi) = 0$ for all $\xi > a$). It follows that $\{\zeta_k\}_{k\in\mathbb{Z}}$ is a basis of $\mathscr{H}_a$.

(c) Fix $f \in \mathscr{H}_a$. By part (b) $\{\zeta_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $\mathscr{H}_a$, so

$$f = \sum\nolimits_{k\in\mathbb{Z}}\langle f|\zeta_k\rangle\zeta_k = \sum\nolimits_{k\in\mathbb{Z}}\langle\widehat{f}|\widehat{\zeta}_k\rangle\zeta_k,$$

where the series converge in L^2. Thus it is is enough to show $\langle f|\zeta_k\rangle = \frac{1}{\sqrt{2a}}f(k/2a)$ for $k \in \mathbb{Z}$ and that $\sum_{k\in\mathbb{Z}} f(k/2a)\zeta_k$ converges to f uniformly. We have

$$\begin{aligned}\langle f^\wedge|\zeta_k^\wedge\rangle &= \frac{1}{\sqrt{2a}}\int f^\wedge(x)e^{2\pi i(k/2a)x}\chi_{[-a,a]}(x)\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2a}}\int_{-a}^a f^\wedge(x)e^{2\pi i(k/2a)x}\,\mathrm{d}x = \frac{1}{\sqrt{2a}}\widehat{\widehat{f}}(-k/2a) = \frac{1}{\sqrt{2a}}f(k/2a),\end{aligned}$$

so it only remains to show the series converges uniformly, and for this it is enough to show the sequence $\{\sum_{k=-N}^{N} f(k/2a)\zeta_k\}_{N\in\mathbb{Z}}$ is uniformly Cauchy.

Fix $\varepsilon > 0$. By Parseval's identity $\sum_{k\in\mathbb{Z}}|\langle f|\zeta_k\rangle| = \|f\|_2^2 < \infty$, so for all sufficiently large $N \in \mathbb{Z}_{\geqslant 0}$

$$\sum_{k\in\mathbb{Z}}|\langle f|\zeta_k\rangle|^2 < \varepsilon. \tag{6.4}$$

Now fix $x \in \mathbb{R}$ and $M, N \in \mathbb{Z}$ with $M \leqslant N$. For all sufficiently large $M, N \in \mathbb{Z}$, we have

$$\left|\sum_{k=M}^{N} f(k/2a)\operatorname{sinc}(2ax-k)\right| = \left|\sum_{k=M}^{N}\langle f|\zeta_k\rangle\zeta_k(x)\right| = \sum_{k=M}^{N}|\langle f|\zeta_k\rangle||\zeta_k(x)|$$
$$\leqslant \left(\sum_{k=M}^{N}|\langle f|\zeta_k\rangle|^2\right)^{1/2}\left(\sum_{k=M}^{N}|\zeta_k(x)|^2\right)^{1/2} \overset{(6.4)}{<} \varepsilon^{1/2}\left(\sum_{k=M}^{N}|\zeta_k(x)|^2\right)^{1/2}$$

where we used the Cauchy–Schwarz inequality. Since $\chi_{[-a,a]}$ is a factor of ζ_k, we may assume $x \in [-a, a]$, and hence that $0 \leqslant |x| \leqslant a$. But we only know this (the previous sentence) for ζ_k, not $\widehat{\zeta_k}$! This requires a correction before the rest of the argument to work. It thus only remains to show the remaining sum term on the right-hand side is uniformly bounded for all $x \in [-a, a]$ as $M, N \to \infty$. For all sufficiently large $M, N \in \mathbb{Z}_{\geqslant 0}$ sufficiently large and $k \in \{M+1, \ldots, N\}$, we have

$$|2ax-k|^2 = |k-2ax|^2 \geqslant \left||k|^2 - 2a|x|\right| \geqslant \frac{|k|^2}{2} = \frac{k^2}{2},$$

and hence

$$\frac{1}{|2ax-k|^2} \leqslant \frac{2}{k^2},$$

so that

$$\sum_{k=M}^{N}|\zeta_k(x)|^2 = \frac{2a}{\pi^2}\sum_{k=M}^{N}\frac{|\sin(\pi(2ax-k))|^2}{|2ax-k|^2} \leqslant \frac{2a}{\pi^2}\sum_{k=M}^{N}\frac{1}{|2ax-k|^2} \leqslant \frac{4a}{\pi^2}\sum_{k=M}^{N}\frac{1}{k^2} < \varepsilon,$$

where the final step is by Folland Exercise 8.13(b). The argument that $\|x \mapsto \sum_{k=M}^{N} f(k/2a)\operatorname{sinc}(2ax-k)\|_u < \varepsilon$ for all sufficiently large $M, N \in \mathbb{Z}$ with $M \leqslant N$ is similar. Thus the series $\sum_{k=-\infty}^{\infty} f(k/2a)\operatorname{sinc}(2ax-k)$ is uniformly Cauchy, and hence converges uniformly.

Lastly, we show $f(x) = \sum_{k\in\mathbb{Z}} f(k/2a)\operatorname{sinc}(2ax-k)$ a.e. and that $f \in C_0$. We already know the partial sums converge to f in L^2, so some subsequence of the partial sums converge to f pointwise a.e., so, after modification of f on a null set f equals the given series. Thus f is the uniform limit of the partial sums—which are themselves continuous since sinc is—so f is continuous. To see f vanishes at infinity, note that if we take the Fourier transformation of Equation (6.3) once more, we obtain

$$\widehat{\widehat{\zeta_k}}(x) = \frac{1}{\sqrt{2a}}\int e^{-2\pi i(k/2a)\xi}\chi_{[-a,a]}(\xi)e^{-2\pi i x\xi}\,\mathrm{d}\xi = \frac{1}{\sqrt{2a}}\int \chi_{[-a,a]}(\xi)e^{2\pi i(-x-k/2a)\xi}\,\mathrm{d}\xi$$
$$= \frac{1}{\sqrt{2a}}\chi_{[-a,a]}^{\vee}(-x-k/2a) \overset{(6.2)}{=} \zeta_k(-x).$$

But $\{\zeta_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $\mathscr{H}_a$, so we have a convergent series in L^2 given by

$$f(-x) = \sum_{k\in\mathbb{Z}} \langle f|\zeta_k\rangle \zeta_k(-x) = \sum_{k\in\mathbb{Z}} \langle f|\zeta_k\rangle \widehat{\widehat{\zeta}}_k(x) = \mathscr{F}^2\Big(x \mapsto \sum_{k\in\mathbb{Z}} \langle f|\zeta_k\rangle \zeta_k(x)\Big)(x) = \widehat{\widehat{f}}(x),$$

where the penultimate equality is by the DCT, so in particular $\widehat{f} \in L^1$ by the Fourier inversion theorem. Thus $\widehat{\widehat{f}}(-x) = f(x)$ is the Fourier transform of an L^1 function, so $f \in C_0$ by the Riemann–Lebesgue lemma. □

Folland Exercise 8.16.

Let $f_k = \chi_{[-1,1]} * \chi_{[-k,k]}$.

(a) Compute $f_k(x)$ explicitly and show that $\|f_k\|_u = 2$.

(b) $f_k^\vee(x) = (\pi x)^{-2} \sin 2\pi kx \sin 2\pi x$, and $\|f_k^\vee\|_1 \to \infty$ as $k \to \infty$. (Use Folland Exercise 8.15(a), and substitute $y = 2\pi kx$ in the integral defining $\|f_k^\vee\|_1$.)

(c) $\mathfrak{F}(L^1)$ is a proper subset of C_0. (Consider $g_k = f_k^\wedge$ and use the open mapping theorem.)

Solution.

(a) Let $[a,b],[c,d] \subset \mathbb{R}$. Then

$$\chi_{[c,d]}(x-y) = \delta_{c \leqslant x-y \leqslant d} = \delta_{x-d \leqslant y \leqslant x-c} = \chi_{[x-d,x-c]}(y),$$

so

$$\begin{aligned}\chi_{[a,b]} * \chi_{[c,d]}(x) &= \int \chi_{[a,b]}(y)\chi_{[c,d]}(x-y)\,\mathrm{d}y = \chi_{[a,b]}(y)\chi_{[x-d,x-c]}(y)\,\mathrm{d}y \\ &= \int \chi_{[a,b]\cap[x-d,x-c]}(y)\,\mathrm{d}y = m([a,b]\cap[x-d,x-c]).\end{aligned}$$

Thus

$$\|f\|_u = \sup_{x\in\mathbb{R}}|m([-1,1]\cap[x-k,x+k])| \leqslant m([-1,1]) = 2.$$

(b) By Folland Exercise 8.15(a),

$$\begin{aligned}f_k^\vee(x) &= (\chi_{[-1,1]} * \chi_{[-k,k]})^\vee(x) = \chi_{[-1,1]}^\vee(x)\chi_{[-k,k]}^\vee(x) \\ &= 2\,\mathrm{sinc}(2x)2k\,\mathrm{sinc}(2kx) = (\pi x)^{-2}\sin(2\pi x)\sin(2\pi kx)\end{aligned}$$

and, making the substitution $y \mapsto 2k\pi x$, we obtain

$$\begin{aligned}\int |f_k^\vee(x)|\,\mathrm{d}x &= \pi^{-2}\int\left|\frac{1}{x^2}\sin(2\pi x)\sin(2\pi kx)\right|\mathrm{d}x \\ &= 4|k|^2\int\left|\frac{1}{y^2}\sin(y)\sin(y/k)\right|\mathrm{d}y = 4|k|\lim_{N\to\infty}\int_{-\infty}^{\infty}\left|\frac{\sin y}{y}\right|\left|\frac{\sin(y/k)}{y/k}\right|\mathrm{d}y.\end{aligned}$$

For all $N \in \mathbb{Z}_{\geqslant 0}$, $\left|\frac{\sin y}{y}\right|\left|\frac{\sin(y/k)}{y/k}\right|\chi_{[N,N]} \leqslant \chi_{[N,N]} \in L^1$, so by the DCT we have

$$\lim_{k\to\infty}\int_{-N}^{N}\left|\frac{\sin y}{y}\right|\left|\frac{\sin(y/k)}{y/k}\right|\mathrm{d}y = \int_{-N}^{N}\lim_{k\to\infty}\left|\frac{\sin y}{y}\right|\left|\frac{\sin(y/k)}{y/k}\right|\mathrm{d}y = \int_{-N}^{N}\left|\frac{\sin y}{y}\right|\mathrm{d}y.$$

Hence

$$\|f_k^\vee\|_1 \geqslant \int_{-N}^{N}\left|\frac{\sin y}{y}\right|\mathrm{d}y \tag{6.5}$$

for all $N \in \mathbb{Z}_{\geqslant 0}$. But the right-hand side diverges to ∞ as $N \to \infty$, which we now show (or, alternatively, by Folland Exercise 2.59). Note that $|\sin x| \geqslant 1/2$ for all $x \in \mathbb{R}$ such that $|x| \in [\pi/6, 5\pi/6], [7\pi/6, 11\pi/6], [13\pi/6, 17\pi/6], \ldots$. On these respective intervals, we have $|1/x| \geqslant 6/5\pi, 6/11\pi, 6/17\pi, \ldots$, and thus $\left|\frac{\sin x}{x}\right| \geqslant 3/5\pi, 3/11\pi, 3/17\pi, \ldots$.

Therefore, for all $k \in \mathbb{Z}_{\geqslant 0}$, by taking the limit of Equation (6.5) as $N \to \infty$, we obtain

$$\int_{-\infty}^{\infty} \left| \frac{\sin y}{y} \right| \mathrm{d}y \geqslant 3\left(\frac{1}{5\pi} + \frac{1}{11\pi} + \frac{1}{17\pi} + \cdots \right) = \frac{3}{\pi} \sum_{N=1}^{\infty} \frac{1}{6N-1} = \infty.$$

(c) Any $\widehat{f} \in \mathfrak{F}(L^1)$ is continuous since $\mathfrak{F}$ maps L^1 into C_0. Now suppose for a contradiction $\mathfrak{F}(L^1) = C_0$. By the Hausdorff–Young inequality, $\mathfrak{F}$ is bounded as a map $L^1 \to C_0$ (since $\widehat{f} \in C_0 \subset C_b$, hence $\|\widehat{f}\|_u = \|\widehat{f}\|_\infty \leqslant \|f\|_1$ for all $f \in L^1$). Thus $\mathfrak{F}$ is a bounded surjection, so by the open mapping theorem $\mathfrak{F}$ is invertible on L^1 and $\mathfrak{F}^{-1} \colon C_0 \to L^1$ is bounded. Then there exists $C > 0$ such that for all $k \in \mathbb{Z}_{\geqslant 0}$,

$$\|\widehat{f}_k\|_1 \leqslant C \|f_k\|_u \overset{\text{(a)}}{=} 2C,$$

contradicting part (b) since $\|\widehat{f}_k\|_1 \to \infty$ as $k \to \infty$. □

Folland Exercise 8.19.

If $f \in L^2(\mathbb{R}^n)$ and the set $S = \{x \mid f(x) \neq 0\}$ has finite measure, then for any measurable $E \subset \mathbb{R}^n$,

$$\|\widehat{f}\chi_E\|_2^2 \leqslant \|f\|_2^2 m(S)m(E).$$

Solution. Given that the measure of the set S is finite ($m(S) < \infty$), it follows that $L^p(S) \subset L^q(S)$ for $1 \leqslant q \leqslant p$. Thus, since $f \in L^2(S)$, we have $f \in L^1(S)$. And for any fixed $\xi \in \mathbb{R}^n$, we have

$$\int_S |e^{2\pi i x\cdot\xi}|^2 \,\mathrm{d}x = \int_S 1 \,\mathrm{d}x = m(S) < \infty,$$

so the map $x \mapsto e^{2\pi i x\cdot\xi}$ is also in $L^2(S)$. Now by Hölder's inequality

$$|\widehat{f}(\xi)| = \left|\int f(x)e^{-2\pi i\xi\cdot x}\,\mathrm{d}x\right| = \left|\int \chi_S(x)f(x)e^{-2\pi i\xi\cdot x}\,\mathrm{d}x\right| \leqslant \|f\|_2\|\chi_S\|_2 = \|f\|_2 m(S)^{1/2}, \quad (6.6)$$

where the second equality is because $f|_{\mathbb{R}^n\setminus S} = 0$ (by definition of S). Thus

$$\|\widehat{f}\chi_E\|_2^2 = \int_E |\widehat{f}(\xi)|^2 d\xi \overset{(6.6)}{\leqslant} \|f\|_2^2 m(S)\int_E 1\,\mathrm{d}\xi = \|f\|_2^2 m(S)m(E). \qquad \square$$

Q5.

Suppose that $f \in L^1(\mathbb{R})$ and both f and $\widehat{f}$ have compact support. Prove that $f = 0$.

Solution. Since we can translate and compose with scalar multiplication, we may assume without loss of generality $\operatorname{supp} f \subset [0, 1/2]$. Since $f \in L^1$, By the Hausdorff–Young theorem $\widehat{f} \in L^\infty$ and $\|\widehat{f}\|_\infty \leqslant \|f\|_1$. Hence $\widehat{f}$ is a.e. bounded, and in particular

$$\|\widehat{f}\|_1 = \int |\widehat{f}| \leqslant \int \|f\|_1 \chi_{\operatorname{supp}(\widehat{f})} < \infty.$$

Thus $\widehat{f} \in L^1$, so by the Fourier inversion theorem f is a.e. continuous and $\widehat{f}^\wedge = (f^\vee)^\wedge = f$.

Since $\operatorname{supp} \widehat{f}$ is bounded, there exists $N \in \mathbb{Z}_{\geqslant 0}$ such that $\widehat{f}(\kappa) = 0$ whenever $|\kappa| \geqslant N$. In particular, the Fourier series of f is $\sum_{m=-N}^N \widehat{f}(m)e^{2\pi imx}$. By a corollary to the Fourier inversion theorem (namely Folland Corollary 8.27), to see $f = \sum_{m=-N}^N \widehat{f}(m)e^{2\pi imx}$ a.e. it suffices to show for $\kappa \in \mathbb{Z}$ that

$$\mathfrak{F}\Big(x \mapsto \sum_{m=-N}^N \widehat{f}(m)e^{2\pi imx}\Big)(\kappa) = \widehat{f}(\kappa).$$

And indeed,

$$\begin{aligned}\mathfrak{F}\Big(x \mapsto \sum_{m=-N}^N \widehat{f}(m)e^{2\pi imx}\Big)(\kappa) &= \int_0^1 \Big(\sum_{m=-N}^N \widehat{f}(m)e^{2\pi imx}\Big)e^{-2\pi i\kappa x}\,dx \\ &= \sum_{m=-N}^N \widehat{f}(m)\int_0^1 e^{2\pi i(m-\kappa)x}\,dx = \sum_{m=-N}^N \widehat{f}(m)\delta_{m,\kappa} = \widehat{f}(\kappa),\end{aligned}$$

so $f = \sum_{m=-N}^N \widehat{f}(m)e^{2\pi imx}$ a.e. But f vanishes on the interval $(1/2, 1)$, so the sum $\sum_{m=-N}^N \widehat{f}(m)e^{2\pi imx} = 0$ must also; but any trigonometric polynomial that vanishes on an interval must be identically zero (e.g., by the identity principle, since trigonometric polynomials are holomorphic), so $f = 0$. □

Q6.

Show that the conditions $\frac{1}{p}+\frac{1}{q}=1$ and $1 \leqslant p \leqslant 2$ in the Hausdorff–Young inequality (Folland Theorem 8.30) are both necessary for such an inequality to hold. [4]

Solution. Suppose $p, q \in [1,\infty]$ satisfy

$$\|\widehat{f}\|_q \leqslant \|f\|_p \text{ for all } f \in L^p(\mathbb{R}^n). \tag{6.7}$$

- *Necessity that the exponents are conjugate*: Suppose $p, q \in [1,\infty]$, and consider an arbitrary $f \in L^p(\mathbb{R}^n)$. For $t > 0$, define $f_t(x) = t^{-n} f(t^{-1}x)$.

$$\|f_t\|_p = \left(\int t^{-np}|f(t^{-1}x)|^p \,\mathrm{d}x\right)^{1/p} = t^{-n}\left(\int t^n|f(x)|^p \,\mathrm{d}x\right)^{1/p} = t^{-n(1-1/p)}\|f\|_p, \tag{6.8}$$

and this equation still holds if $p = \infty$ with the convention $1/p = 0$. Now in particular we know $f_t \in L^p$. Now write

$$\widehat{f}_t(\xi) = t^{-n}\int f(t^{-1}x)e^{-2\pi i\xi\cdot x}\,\mathrm{d}t = \int f(y)e^{-2\pi i\xi\cdot(y/t)}\,\mathrm{d}y = \widehat{f}(t\xi).$$

Then

$$\|\widehat{f}_t\|_q = \left(\int |\widehat{f}(t\xi)|^q \,\mathrm{d}\xi\right)^{1/q} = t^{-n/q}\left(\int |\widehat{f}(\xi)|^q \,\mathrm{d}\xi\right)^{1/q} = t^{-n/q}\|\widehat{f}\|_q \tag{6.9}$$

so

$$\|\widehat{f}\|_q \overset{(6.9)}{=} t^{n/q}\|\widehat{f}_t\|_q \overset{(6.7)}{\leqslant} t^{n/q}\|f_t\|_p \overset{(6.8)}{=} t^{n/q}t^{-n(1-1/p)}\|f\|_p = t^{n\left(\frac{1}{p}+\frac{1}{q}-1\right)}\|f\|_p,$$

where we use the condition that $1/q = 0$ for $q = \infty$. But $t > 0$ was arbitrary, so this must hold for all such t; thus $1/p + 1/q - 1 = 0$, so p and q are conjugate exponents. Thus the conjugate exponent condition in the Hausdorff–Young inequality is necessary for $p, q \in [1,\infty]$.

- *Necessity that $p \in [1,2]$*: Suppose for a contradiction $p \in (2,\infty]$ and again consider an arbitrary $f \in L^1(\mathbb{R}^n)$. First note $p \neq \infty$, since otherwise by Folland Exercise 8.15 the $L^1(\mathbb{R})$ function $\chi_{[-\frac{1}{2},\frac{1}{2}]}$ satisfies

$$\infty = \int_{-\infty}^{\infty}\left|\frac{\sin(\xi)}{\xi}\right|\mathrm{d}\xi = \|\widehat{\chi}_{[-\frac{1}{2},\frac{1}{2}]}\|_1 \overset{(6.7)}{\leqslant} \|\chi_{[-\frac{1}{2},\frac{1}{2}]}\|_\infty = 1,$$

a contradiction (and the case of general $n \in \mathbb{Z}_{\geqslant 1}$ is similar by considering $\chi_{[-1/2,1/2]^n}$), so we may assume $p \in (2,\infty)$.

Let $f_s(x) = s^{-n/2}e^{-\pi|x|^2/s}$ and let $h(x) = e^{-\pi s|x|^2}$, so that $f_s = \widehat{h}$ by Folland Proposition 8.24. By our assumption (6.7) and the previous point, $1/p + 1/q = 1$. Then $q \in (1,2)$, and in particular $q < p$. We have

$$\|h\|_p = \left(\int |e^{-\pi s|x|^2}|^p \,\mathrm{d}x\right)^{1/p} = \left(\int e^{-\pi p|x|^2}\,\mathrm{d}x\right)^{1/p} \overset{\text{Folland Prop. 2.53}}{=} p^{-n/2p} \tag{6.10}$$

and

$$\|\widehat{h}\|_q = \|f_s\|_q = \left(\int |s^{-n/2}e^{-\pi|x|^2/s}|^q\right)^{1/q} = |s|^{-n/2}\left(\int e^{-\pi q(1+t^2)^{-1}|x|^2}\,\mathrm{d}x\right)^{1/q}$$
$$\overset{\substack{\text{Folland}\\ \text{Prop. 2.53}}}{=} |s|^{-n/2}\left(\frac{\pi}{\pi q(1+t^2)^{-1}}\right)^{n/2q} = (1+t^2)^{-n/4}q^{-n/2q}(1+t^2)^{n/2q}$$
$$= q^{-n/2q}(1+t^2)^{\frac{n}{4}(\frac{2}{q}-1)} = q^{-n/2q}(1+t^2)^{\frac{n}{4}\left(\frac{1}{q}-\frac{1}{p}\right)}, \tag{6.11}$$

where for the last equality we used the requirement from the previous point that $1/p + 1/q = 1$. In particular $h \in L^p(\mathbb{R}^n)$, so by our assumption (6.7)

$$p^{-n/2p} \overset{(6.10)}{=} \|h\|_p \geqslant \|\widehat{h}\|_q \overset{(6.11)}{=} q^{-n/2q}(1+t^2)^{\frac{n}{4}\left(\frac{1}{q}-\frac{1}{p}\right)}.$$

Raising both sides to the power of $-2/n$, we obtain

$$p^{1/p} \leqslant q^{1/q}(1+t^2)^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}. \tag{6.12}$$

Since $p < q$ by assumption, $-1/2(1/q - 1/p) < 0$, so by choosing $t > 0$ appropriately we can make $(1+t)^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}$ arbitrarily small. But $p^{1/p}$ is strictly positive, so this contradicts Equation (6.12). Thus $p \notin (2, \infty]$, so $p \in [1, 2]$. □

Alphabetical Index