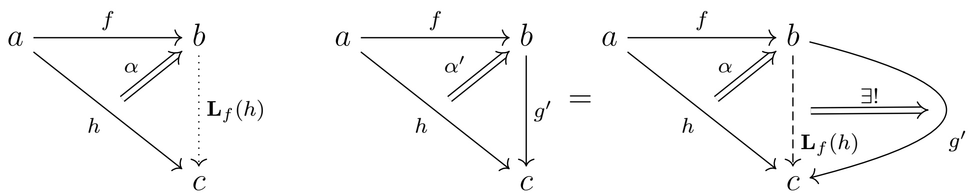
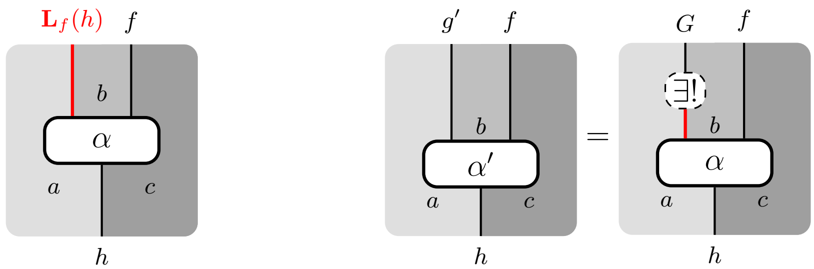
Definition 1.0.1 ([Rie16, Definition 6.1.1, altered])**.** For a 2-category $\mathcal{C}$ and 1-morphisms $h \in \mathcal{C}(a \to c)$, $f \in \mathcal{C}(a \to b)$, a *left (Kan) extension* of h along f is a pair (g, α) consisting of a 1-morphism $g \in \mathcal{C}(b \to c)$ and a 2-morphism $\alpha\colon h \Rightarrow f \otimes g$ in $\mathcal{C}$ such that any other such pair $(g', \alpha'\colon h \Rightarrow f \otimes g')$ factors uniquely through α as illustrated. We typically write $\mathbf{L}_f(h)$ for g.

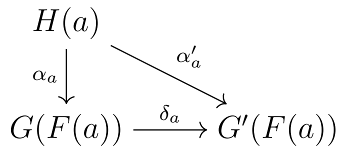


Using the graphical calculus for 2-categories with the convention $\otimes$ reads right-to-left, these pasting diagrams become



A *right (Kan) extension* in $\mathcal{C}$ is a left Kan extension in $\mathcal{C}^{2\mathrm{op}}$, the 2-category obtained from $\mathcal{C}$ by reversing all the 2-morphisms.

Example 1.0.2. When $\mathcal{C} = 2\mathsf{Cat}$, in which case we capitalize all roman letters in the above equation and set $G := \mathbf{L}_f(h)$ for clarity, the equation on the right means that there is a unique natural transformation $\delta\colon G \Rightarrow G'$ such that for all $a \in A$, the following diagram commutes.



We write $\mathbf{1}$ for the category with exactly one object 1 and exactly one morphism, namely $\mathrm{id}_1\colon 1 \to 1$.

Example 1.0.3. The colimit of a functor $F\colon \mathcal{C} \to \mathcal{D}$ is the value at 1 of the left Kan extension of F along the functor $\mathcal{C} \to \mathbf{1}$. That is,

$$\operatorname{colim} F = (\mathbf{L}_{\mathcal{C}\to\mathbf{1}}(F))_1.$$

For example, suppose F is a diagram in Top of shape $* \leftarrow \bullet \rightarrow \star$, say $X \xleftarrow{f} Y \xrightarrow{g} Z$, and $\sim$ is the equivalence relation determined by the relations $f(y) \sim g(y)$ for all $y \in Y$, so that $\operatorname{colim} F = (X \amalg Y)/{\sim}$.

The unique functor from the index category $X \xleftarrow{f} Y \xrightarrow{g} Z$ to $\mathbf{1}$ TODO:

References

[Rie16] Emily Riehl. *Category Theory in Context*. Dover Publications, 2016.