Graph Tools and Polynomials

Graph Tools: $f(x) \mapsto f(-x)$ reflect ab y-axis. $f(x) \mapsto -f(x)$: reflect ab x-axis. $f(x) \mapsto -f(-x)$:reflect ab $x$and$y$-axes (180°rot ab origin). $f(x) \mapsto f^{-1}(x)$: reflect ab $y{=}x$. **Polynomials: Finding roots:** Try ± 1. Let r be a root. Divide by $x-r$ to find more. $f \in \mathbb{Z}[x] \Rightarrow$ rational roots $\in \{\pm_{\text{divisors of}} a_0\}/\{\pm_{\text{divisors of}}\ a_n\}$. **Vieta**: $\Sigma_{\text{roots}} = -a_{n-1}/a_n$; $\Pi_{\text{roots}} = (-1)^n a_0/a_n$. $f(r) = 0 \Leftrightarrow \lambda f(r) = 0$. r of multip. $k \Leftrightarrow f(r),\ldots,f^{(k-1)}(r) = 0$, $f^{(k)}(r) \neq 0$. f monic $\Rightarrow |r| < \max(|a_i|) + 1$. **Descartes' Rule of Signs:** # of *positive* roots of $f = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is the # sign changes of the consecutive nonzero coefficients of $f(x)$ (reading from left to right), or an even number less; # of *negative* roots of f is the # of sign changes of $f(-x)$, or an even number less. (e.g. $f(x) = 2x^5 - 8x - 7$ has 1 sign change so must have one positive root, and $f(-x) = -2x^5 - 8x + 7$ has 0 sign changes, so must have zero negative roots, so f has 1 real root in total, the rest being complex.)

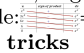
Trigonometric Identities

sin odd, cos even. $\sin(\pi/2 - x) = \cos x$. $\cos(\pi/2 - x) = \sin x$. $\tan(\alpha \pm \beta) = (\tan\alpha \pm \tan\beta)/(1 \mp \tan\alpha\tan\beta)$. $\tan(2x) = 2\tan x/1 - \tan^2 x$. $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$. $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$. $\sin x = \tan x/\sqrt{1 + \tan^2 x}$. $\cos x = 1/\sqrt{1 + \tan^2 x}$. $\partial_x(x\log x - x) = \log x$. $\partial_x \log|\tan(x/2)| = \csc x$.

Geometry/Precalculus/Some Differential Calculus

Pyramid/ConeVol: $Bh/3$,Bbase area(e.g.$\pi r^2 h/3$). Trapezoid vol: $(a+b)h/2$. Hero's formula: $\triangle abc$, $s = (a+b+c)/2$, then $A = \sqrt{s(s-a)(s-b)(s-c)}$. Law of cosines: $c^2 = a^2 + b^2 - 2ab\cos(C)$. Law of sines: $a/\sin(A) = b/\sin(B) = c/\sin(C) = 2R$, R radius of circumscribed circle. Circumscribe circle of shape S =circle s.t. vertices of S lie on circumference. Inscribed circle of S: circle tangent to each edge of S. $x^2 + y^2 = L^2 \Rightarrow x(dx/dt) + y(dy/dt) = 0$. $\dot{V}_{\text{sphere}} = A_{\text{sphere}}$. $\log_b x = \log_a x/\log_a b$; $\log_b a = 1/\log_a b$. $\partial_x \arcsin(x) = 1/\sqrt{1-x^2} = -\partial_x \arccos(x)$. $\partial_x \arctan(x) = 1/1 + x^2$. $\partial_x \operatorname{arcsec}(x) = 1/(|x|\sqrt{x^2-1})$. If f is hard to differentiate and $\log(f)$ is easier to differentiate, use that $f' = (\log(f))'f$. Tools for finding $\lim F$ include l'Hôpital's Rule, Taylor expansion, or e^L, where $L = \lim(\log(F))$.

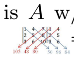
Integral Calculus

$\int \tan = -\log|\cos|$. $\int \cot = \log|\sin|$. $\int \sec = \log|\sec + \tan|$. $\int \csc = -\log|\csc + \cot|$. $\int 1/(a^2 + x^2)dx = a^{-1}\arctan(u/a)$. $\int 1/\sqrt{a^2 - u^2} = \arcsin(u/a)$. If something has the form $f(x) - f(0)$, think of setting up integral $\int_0^x f'$. $\sqrt{a^2 - x^2}$: $x = a\sin\theta$. $\sqrt{a^2 + x^2}$:$x = a\tan\theta$. $\sqrt{x^2 - a^2}$:$x = a\sec\theta$. **Polar**: Area from $\theta \in [a,b]$ of $r = f(\theta)$: $\frac{1}{2}\int_a^b f(\theta)^2 d\theta$. **Arc length** from $x \in [a,b]$: $\int_a^b \sqrt{1 + f'(x)^2}dx$, or from $t \in [a,b]$: $\int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}dt$. Volume rotate $f \leqslant g$ about x-axis from $x \in [a,b]$:$\pi\int_a^b (g(x)^2 - f(x)^2)dx$. **Int. by parts preference for what u is**: ILATE (inverses, logarithms, algebraic (i.e. polynomials/rational functions), trig, exponentials). **Tabular method of int. by parts**: (left column is u [decided w/ ILATE] and successive derivatives; right column is dv and its successive primitives. Example:  $\Rightarrow$ $\int x^3 e^x dx = x^3e^x - 3x^2e^x + 6xe^x - 6e^x$/ **Integral tricks if stuck:** $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ (so that $\int_a^b f(x)dx = \frac{1}{2}\int_a^b (f(x) + f(a+b-x))dx$). f even$\Rightarrow \int_{-a}^a \frac{f(x)}{1+e^x}dx = \int_0^a f(x)dx$. $\int f(\sin x, \cos x)dx = \int f(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2})\frac{2}{1+t^2}dt$, $t = \tan(x/2)$. **Differentiation under the integral sign (Feynman's trick):** If you want $I(\alpha) = \int_a^b F(x,\alpha)dx$ (usually α is a constant, e.g. 1) and $I'(\alpha) = \int_a^b F_\alpha(x,\alpha)d\alpha$ is easier to integrate, then $I(\alpha) = \int I'(\alpha)d\alpha$. If $\int_a^\infty \frac{f(\alpha x) - f(x)}{x}dx$ cvgs $\forall a > 0$ and $\lim_{x\to 0} f(x) = a$ then $\int_0^\infty \frac{f(\alpha x) - f(x)}{x}dx = A\log(\alpha)$. $\int_{-\infty}^\infty f(x)dx = \frac{1}{2}\int_{-\infty}^\infty (f(x) + f(-x))dx = \int_0^\infty (f(x) + f(-x))dx$. **Series Convergence Tests: Geometric Series Test:** Σx^n cv $\Leftrightarrow |x| < 1$. **p-Series Test:** $\Sigma 1/n^p$ converges $\Leftrightarrow p > 1$. **Ratio Test:** $r := \lim|a_{n+1}/a_n|$ then: $r < 1 \Rightarrow \Sigma a_n$ cnverges (absolutely), $r > 1 \Rightarrow \Sigma a_n$ divgs, $r = 1 \Rightarrow$ inconclusive. **Root Test:** $r := \lim|a_n|^{1/n}$ then $(*)$. **Integral Test:** f cts, $\geqslant 0$, decreasing on $[1,\infty)$ then $\Sigma f(n)$ cvg $\Leftrightarrow \int_1^\infty f$ cvg. Alternating Series Test: $a_n \to 0 \Rightarrow \Sigma(-1)^n a_n$ cvg. **Cauchy Condensation Test:** $a_n \geqslant 0$, $a_n \to 0$ then $\sum a_n$ cvgs $\Leftrightarrow \sum a_{2^n}2^n$.

Expansions/Approximations

$1/(1-x) = 1 + x + x^2 + x^3 + \cdots$. $e^x = 1 + x + x^2/2! + x^3/3! + \cdots$. $\log(\mathbf{1}{+}x) = x - x^2/2 + x^3/3 - \cdots$. $\sin x = x - x^3/3! + x^5/5! - \cdots$. $\cos x = 1 - x^2/2! + x^4/4! - \cdots$. $\arctan(x) = x - x^3/3 + x^5/5 - \cdots$. $\arctan(x) = \pi/2 - \arctan(1/x)$. $\sqrt{1+x} = 1 + x/2 - x^2/8 + x^3/16 - \cdots$. $n! \sim \sqrt{2\pi n}(n/e)^n$. $(1+x)^r \approx 1 + rx + \frac{r(r-1)}{2}x^2 + \frac{r(r-1)(r-2)}{6}x^3$. $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^n \Delta x f(a + i\Delta x)$, where $\Delta x = \frac{(b-a)}{n}$ (usually $a = 0, 1$ and $\Delta x = 1/n$ or something simple).

Linear Algebra

$m \times n$ matrix E is in **row echelon form** if (1) all zero rows are below all nonzero rows, and (2) the first nonzero entry of a row is in a column strictly to the right of that of the first nonzero entry of any previous row. If E in echelon form then we say E is in **reduced row echelon form** if (3) any first nonzero entries are 1, and (4) each column containing a leading 1 has zeros in all its other entries. The **row echelon form of** A, A_{e} (resp. the reduced row echelon form of A), denoted A_{e} (resp. A_{re}), is any matrix in row echelon form (resp. reduced row echelon form) obtained by **elementary row operations**, which are (T_1) scalar multiplication of a row, (T_2) swapping any two rows, or (T_3) adding a scalar multiple of one row to another row. A_{e} is *not unique*, while A_{re} is *unique.* A **system of eqns** $[A|b]$ is **consistent** if $\exists \geqslant 1$ soln $\mathbf{x}$. A system of eqns $[A|b]$ is **inconsistent** if $\nexists$ soln $\mathbf{x}$. $A\mathbf{x} = \mathbf{b}$ *in*consistent $\Leftrightarrow$ last column of echelon form of augmented matrix $[A|\mathbf{b}]$ has pivot. Cols of A form a basis $\Leftrightarrow$ *all rows and all columns* of A_{e} have pivot $\Leftrightarrow \exists$ *unique* soln for $A\mathbf{x} = \mathbf{b} \Leftrightarrow A$ invertible. Cols of A spanning $\Leftrightarrow A$ surj $\Leftrightarrow$ every *row* of A_{e} has pivot. Cols of A l.i. $\Leftrightarrow A$ inj $\Leftrightarrow$ every *column* of A_{e} has pivot. **Matrix properties**: A is $m \times n$ & B is $n \times p$, then AB is $(m \times n)(n \times p) = m \times p$ (think of inner ones vanishing, e.g. n in this case). A, B square and A left(or right) invertible w/ left inverse B then B is the *full* inverse of A (i.e. then $AB = BA = I$). $A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \Rightarrow A^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a\end{smallmatrix}\right)$. $\text{ch}(A){=}l^2 - l\,\text{tr}(A){+}\det(A).\Sigma l_i = $ tr. $\Pi l_i = $ det. **Cramer's Rule**: A square, $A\mathbf{x} = \mathbf{b} \Rightarrow x_i = \det A_j/\det A$, where A_j is A w/ col j replaced by $\mathbf{b}$. **Determinant trick for any** 3×3:  $\Rightarrow$ add blue, subtract red. $A^n = 0 \Rightarrow (I - A)^{-1} = I + A + \cdots + A^{n-1}$. **The four fundamental subspaces** are $\text{im}(A)$ (column space), $\ker(A)$ (nullspace), $\text{im}(A^t)$ (row space), $\ker(A^t)$. **Finding bases for row/col(img)/ker spaces: Basis for col space** $\text{im}(A)$ is the cols A_j $_{\text{of } A}$ s.t. the corresponding $(A_{\text{e}})_j$ have pivot. **Basis for row space** $\text{im}(A^t)$ is rows of A_{e} with pivot (or j/ use basis of the col space of A^t). Basis for nullspace $\ker(A)$ is solution set of linear eqns $A\mathbf{x} = 0$. $\nexists$ any shortcut for the subspace $\ker(A^t)$; you need to find basis for $\mathbf{x}$ s.t. $A^t\mathbf{x} = 0$.

Affine and Analytic Geometry

$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}$. $\text{proj}_{\mathbf{a}}(\mathbf{b}) = \mathbf{a}\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}})\mathbf{a}$. Vector *from* $\mathbf{a}$ *to* $\mathbf{b}$ (i.e. head at $\mathbf{a}$ and tail $\mathbf{b}$) is $\mathbf{a} - \mathbf{b}$. **Line through $\mathbf{p}$, $\mathbf{q}$** is $t\mathbf{p} + (1-t)\mathbf{q}$. **Plane through $\mathbf{x}_0$ normal to $\mathbf{n}$** is $\mathbf{n}\cdot(\mathbf{x} - \mathbf{x}_0) = 0$, i.e. $n_1x + n_2y + n_3z = \mathbf{n}\cdot\mathbf{x}_0$. **Plane through 3 pts $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$** is $\mathbf{v}\cdot\mathbf{x} = d$, where $\mathbf{v}, d$ satisfy $\mathbf{v}\cdot\mathbf{x}_1 = d$, $\mathbf{v}\cdot\mathbf{x}_2 = d$, $\mathbf{v}\cdot\mathbf{x}_3 = d$ (expand dot products and solve w/ row-reductn). **Graphing Tools**: Cylinder: $x^2 + y^2 = a^2$. Cone: $x^2 + y^2 = z^2$. Sphere: $x^2 + y^2 + z^2 = a^2$. We can substitute $x^2 + y^2 \to r$ to reduce to a two-dimensional graph of z (or whatever variable is not involved) and r, by "unwrapping" the 3D graph around the z-axis. Circles around the z-axis are then level-curves, too.

Multivariable Differentiation:

Tangent plane of graph $z = f(x,y)$ at $(x_0, y_0, z_0 = f(x_0,y_0))$: $z - z_0 = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$. **Partial Derivative Chain Rule:** If $u = f(x,y)$, $v = g(x,y)$ then partial derivatives of $F = F(u,v)$ are $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x}$, and simil. for $\frac{\partial F}{\partial y}$. $f{:}\mathbb{R}^2{\to}\mathbb{R}$. Then $\nabla f = (f_x, f_y)$, so approx near (x_0,y_0) is $f(x,y) \approx f(x_0,y_0) + \nabla f\cdot\mathbf{x}$. Direction of maximized slope is the direction $\hat{\mathbf{v}} = \nabla f/\|\nabla f\|$ (with slope $\|\nabla f\|$). Then the minimal slope is $-\hat{\mathbf{v}}$ (with slope $-\|\nabla f\|$). **Local Extrema:** (1) Find $\mathbf{p}$ s.t. $\nabla f(\mathbf{p}) = 0$ (the crit. pts). (2) Determine Hessian for each $\mathbf{p}$: $H_f(\mathbf{p}) = f_{xx}(\mathbf{p})f_{yy}(\mathbf{p}) - f_{xy}(\mathbf{p})^2$. (3) $H_f(\mathbf{p}) > 0$ and $f_{xx}(\mathbf{p}) > 0$ (resp. $f_{xx}(\mathbf{p}) < 0$) then $\mathbf{p}$ is local min (resp. local max). $H_f(\mathbf{p}) < 0 \Rightarrow \mathbf{p}$ a sattle point. $H_f(\mathbf{p}) = 0 \Rightarrow$ inconclusive. **Global Extrema:** (1) Find local extrema. (2) Check the boundaries (usually single-variable functions). If $f(x,y)$ is linear (i.e. $ax + by + cz = d$) then extrema are always on the boundaries. **Extrema Subject to Constraint:** Find min/max of $f(x,y)$ such that $g(x,y) = 0$: $F(x,y,l) := f(x,y) - \lambda g(x,y)$. Then crit. pts. $\mathbf{p}$ $_{\text{for } F}$ are the candidate extrema (i.e. find z in trms of x, y, then solve the system $F_x(x,y,l) = F_y(x,y,l) = g(x,y) = 0$). Then compare values.

Multivariable Integration

Line Integral of the First Kind: Gets area under $z = f(x,y)$ and above a curve C parameterized by $g(x(t), y(t)) = 0$ for $t \in [a,b]$: $\int_a^b f(x(t), y(t))\sqrt{\dot{x}^2(t) + \dot{y}^2(t)}dt$. **Line Integral of the Second Kind:** Gets the net work done by the vector field $\mathbf{F}(x,y) = (P(x,y), Q(x,y))$ on a particle moving along a curve C parameterized by $\mathbf{r} = (x(t), y(t))$, $t \in [a,b]$: $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b (P(x(t)) + Q(y(t))dt (= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C Pdx + Qdy)$. *Direction matters here* but not the specific param. E.g. C unit circle oriented ccw by $\mathbf{r} = (\cos t, \sin t) \Rightarrow \int_C xdx + ydy = \int_0^{2\pi}(\cos t(-\sin t) + \sin t(\cos t))dt = 0$. **FTOC for Line Integrals of the Second Kind:** If $\mathbf{F}(x,y)$ is gradient of g (i.e. $\mathbf{F}$ is **gradient field**), $\int_{C\mathbf{a}\to\mathbf{b}} \mathbf{F}\cdot d\mathbf{r} = g(\mathbf{b}) - g(\mathbf{a})$. **Green's Theorem:** A bounded by closed curved C, simple (no self-intersct), ccw-orient, $P, Q \in C^1(A) \Rightarrow \oint_C Pdx + Qdy = \int_A (Q_x - P_y)dA$. **Divergence/Gauss Thm:** Divergence over volume = surface flux, i.e. $\int_V \nabla\cdot\mathbf{F}dV = \oint_A \mathbf{F}\cdot\hat{\mathbf{n}}ds$. **Stokes' Thm:** Domain D pos. oriented. etc, then $\oint_{\partial D}\mathbf{F}\cdot d\mathbf{r} = \int\cdots\int_D \nabla\mathbf{F}dV$. **Stokes' Thm (Curl Theorem):** The line integral of a vector field over a loop is equal to the flux of its curl through the enclosed surface, i.e. $\iint_A \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}ds = \oint \mathbf{F}d\ell$ **Gradient Field Char.** $\mathbf{F} \in C^1(R)$, R smply cnctd $\Rightarrow \mathbf{F}$ gradient field $\Leftrightarrow P_y = Q_x \Leftrightarrow$ line int. of $\mathbf{F}$ is path-independent $\Leftrightarrow$ line int. around any closed path is zero. Green's theorem: area of A is $\oint_C xdy = -\oint ydx = \frac{1}{2}\oint(xdy - ydx)$.

Numerical Analysis Topics

Let $(x_1,y_1),\ldots,(x_N,y_N)$ be pts in plane. **Center of Gravity**: $X = \frac{1}{N}\Sigma x_i, Y = \frac{1}{N}\Sigma y_i$. **Least Squares Linear Regression** finds a, b s.t. line $y = ax + b$ minimizes y-error $\Sigma(y_i - (ax_i + b))^2$. Goes through center of gravity. **Methods for Finding Solns** $f(x) = 0$, **f cts: Bisection Method**: find pts w/ $f > 0$, $f < 0$, bisect them, try w new pt. **Newton's Method**: Converges faster than bisection: f is differentiable then Newton's method: guess x_0, set $x_n := x_{n-1} - f(x_{n-1})/f'(x_{n-1})$. **Algorithms:**'if':piecewise.

Graph Theory

Hamiltonian Walk (resp. cycle): A walk (resp. cycle) going through each *vertex* exactly once. **Eulerian Walk (resp. cycle)**: A walk (resp. cycle) going through each *edge* exactly once. Eulerian cycle $\Leftrightarrow$ all vertex degrees are even. Eulerian walk $\Leftrightarrow$ all or $n-2$ vertex degrees are even. $\sum_{v\in V}\deg(v) = 2|E|$. T a tree $\Leftrightarrow |V(T)| = |E(T)| + 1$.

 Made by Greyson C. Wesley.

Set Theory and Logic

$A \subset X \Rightarrow f(A) \subset Y$. $B \subset Y \Rightarrow f^{-1}(B) \subset X$. $f(A_1 \cup A_2) = f(A) \cup f(A_2)$. $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$ (equality if f inj). $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$. $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$. $f: A \twoheadrightarrow B$ surjective $\Rightarrow |A| \geqslant |B|$. $g: A \rightarrowtail B$ injective $\Rightarrow |A| \leqslant |B|$. $h: A \to B$ bijection $\Rightarrow |A| = |B|$. If f, g exist as above then $|A| = |B|$. If $A \rightarrowtail B$ and $B \rightarrowtail A$ then $\exists$ a bijection. **Cantor**: $|\mathcal{P}(A)| > |A|$. Ctbl union of ctbl sets is ctbl(e.g.$\mathbb{Q} \cong \cup_{j \in \mathbb{Z}} \mathbb{Z}/j$). Finite cartesian product of ctbl sets is ctbl(e.g.$\mathbb{Q} \times \mathbb{Q}$). A ctbl $\Rightarrow A[x]$ ctbl. Algebraic #s are ctbl. $\mathcal{P}$(ctbl set)=unctbl (bc not in bijection w/ n by Cantor's thm). arctan is a bijection from $\mathbb{R} \to (-\pi/2, \pi/2)$. Modifying this gives bijection from $\mathbb{R}$ to any interval. **Logic:** $P \to Q$ equiv. to $\neg P \vee Q$. $P \to Q$ is false iff P is true and Q is false simultaneously.

Counting

Multinomial Coef:$\binom{n}{k_1, \ldots, k_r} = \frac{n!}{k_1! k_2! \cdots k_r!}$: # ways to put n labld balls in r labld containers, $C_1, \ldots, C_r$, where C_i has exactly k_i elements. $[a^i b^j c^k](a+b+c)^n = \binom{n}{i,j,k}$.

Counting Table

k items	n slots	No Restrictions	$\leqslant$1 items/slot	$\geqslant$1 items/slot
Lbl.	Lbl.	n^k	$k!\binom{n}{k}$	$k!\binom{k}{n}$
Unl.	Lbl.	$\binom{n+k-1}{k}$	$\binom{n}{k}$	$\binom{k-1}{k-n}$
Lbl.	Unl.	$\binom{k}{1}+\cdots+\binom{k}{n}$	$[k \leqslant n]$	$\binom{k}{n}$
Unl.	Unl.	$p(k,1)+\cdots+p(k,n)$	$[k \leqslant n]$	$p(k,n)$

Probability

X a set, F a family of subsets of X is a **Boolean algebra** if $X, \varnothing \in F$ and $A, B \in F \Rightarrow A \cap B, A \cup B, A^c \in F$. A **probability function** is $\mathbb{P}: F \to [0,1]$ s.t. $\mathbb{P}(X) = 1$, $\mathbb{P}(\varnothing) = 0$, $\mathbb{P}(A \sqcup B) = \mathbb{P}(A) + \mathbb{P}(B)$. $\mathbb{P}(A^c) = 1 - P(A)$. Events A, B are **independent** if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. A pt $x \in X$ is **outcome/simple event**. If # possible events is ctbl then say X is **discrete**. X discrete$\Rightarrow \mathbb{P}(A) = |A|/|X|$ =(#outcomes in A)/(#total outcomes). A **random variable** is a function $V: X \to \mathbb{R}$. For **discrete random vars: Expectation (mean):** $\mathbb{E}(Y) = \Sigma y \mathbb{P}(Y = y)$. **Variance:** $\mathbb{V}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 \geqslant 0$. FOr **non-discrete random vars** Y: A **distribution function** is $F_Y(t) = \mathbb{P}(Y \leqslant t) = \int_{-\infty}^t f_Y$, where $f_Y = F_Y'$ is **probability density function**. $\mathbb{E}(Y) = \int_{-\infty}^{\infty} s f_Y(s) ds$, $\mathbb{V}(Y)$ =no change. **Bernoulli Random Vars:** Either 0 or 1. 1 if experiment with probability p is success, 0 otherwise. Exmples: one coin flip, one random binary digit, asking whether a disk drive crashed. $\mathbb{P}(X = 1) = p$, $\mathbb{P}(X = 0) = (1-p)$. $\mathbb{E}(X) = p$. $\mathbb{V}(X) = p(1-p)$. **Binomial random vars** represent the # of successes in n successive indep. trials of a Bernoulli experiment. Exmples: the # of heads from n coin flips, the # of disk drives that crashed in a cluster of 1000 computers, etc. and # of advertisements that are clicked when 400 are served. $\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$ if $k \in \mathbb{N}$, $0 \leqslant k \leqslant n$ (0 otherwise). $\mathbb{E}(X) = np$. $\mathbb{V}(X) = np(1-p)$. Exmpl: X = # heads appearing in 3 coin flips$\Rightarrow X$ = binom random var, $p = 0.5$. What is the probability of each of the different values of X?: $\mathbb{P}(X = 0) = \binom{3}{0} p^0 (1-p)^3 = \frac{1}{8}$, $\mathbb{P}(X = 1) = \binom{3}{1} p^1 (1-p)^2 = \frac{3}{8}$, $\mathbb{P}(X = 2) = \binom{3}{2} p^2 (1-p)^1 = \frac{3}{8}$, $\mathbb{P}(X = 3) = \binom{3}{3} p^3 (1-p)^0 = \frac{1}{8}$. Exmpl of non-discrete random-var: A **normally distributed random variable** Y has $\mathbb{E}(Y) = \mu$, $\mathbb{V}(Y) = \sigma^2$, the **normal dist.** is $f_Y(s) = e^{-((s-\mu)/\sqrt{2}\sigma)^2} / \sigma\sqrt{2\pi}$. $\mathbb{P}(Y < \mu) = 0.5$. $\mathbb{P}(Y < \mu + \sigma/2) \approx 0.691$. $\mathbb{P}(Y < \mu + \sigma) \approx 0.841$. $\mathbb{P}(Y < \mu + 2\sigma) \approx 0.977$. $\mathbb{P}(Y + 3\sigma) \approx 0.999$. The *Gaussian* (bell curve) is symmetric about the line $x = \mu$, so e.g. $\mathbb{P}(Y < \mu - \sigma) = \mathbb{P}(Y > \mu + \sigma) \approx 1 - 0.841 = 0.159$. It is **standard normal dist.** when $\mu = 0$, $\sigma = 1$. The ***z*-score** is # std. devs from the mean, i.e. $z = (x - \mu)/\sigma$, where x is observed value. The ***t*-score** is $t = (\overline{x} - \mu)/(s/\sqrt{n})$, where $\overline{x}$ is mean of sample (i.e. $(\sum x_i)/n$), s=std. dev. of sample, $s = \sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}/\sqrt{n}$.

Number Theory

$\gcd(a,b)\,\mathrm{lcm}(a,b) = ab$. **Euclidean Algorithm**: $a = bq + r \Rightarrow \gcd(a,b) = \gcd(b,r)$. $\gcd(x,0) = x$. **Modular Arithmetic:** p prime. $a^p \equiv a \pmod p$ for any a. If $a \neq kp$ then $a^{p-1} \equiv 1 \pmod p$. If $\gcd(a,n) = 1$ then $a^{\varphi(n)} \equiv 1 \pmod n$. $\varphi(n) = n\Pi_{p|n}(p-1)/p$. **Chinese Remainder Theorem**: a, b coprime & $n_1 p + n_2 q = 1$ and $\begin{cases} x \equiv a \pmod{n_1} \\ x \equiv b \pmod{n_2} \end{cases}$ has soln. $x \equiv a(n_2 q) + b(n_1 p) \pmod{n_1 n_2}$. **Congruence Equation** $ax \equiv b \pmod n$: $ax \equiv b \pmod n$ has soln $\Leftrightarrow$ $\gcd(a,n)$ divides b. If a, n coprime then soln. is unique mod n, otherwise soln. unique mod$(n/\gcd(a,n))$. If looking for $b^n \pmod{10}$, just compute cycle of last digit when multiplying by b, e.g. $7^{25} = (7^5)^5$ and $7^5 \equiv 7$ (mod 1)0, so has last digit 7.

Group Theory

$G = \langle \alpha \rangle$ finite cyclic $\Rightarrow |\langle \alpha^m \rangle| = n/\gcd(m,n)$. **Lagrange's Theorem:** G finite, $H < G \Rightarrow |H|$ divides $|G|$. G finite, *abelian* $\Rightarrow G$ has a subgroup of order d for all divisors d of n. G finite, *cyclic* $\Rightarrow G$ has exactly one (cyclic) subgroup for every divisor d of n. **Cauchy's Theorem**: G finite of order n, p prime dividing $n \Rightarrow G$ has a subgroup of order p. **First Sylow Theorem:** For any finite group G and prime p, if p^k divides $|G|$, then G has a subgroup of order p^k. In other words, if G finite of order $n = p^k m$, $p \nmid n$ $\Rightarrow G$ has subgroup of order p^i for all $0 \leqslant i \leqslant k$. For cyclic groups, $C_m \oplus C_n$ is cyclic $\Leftrightarrow \gcd(m,n) = 1$, in which case $C_m \oplus C_n \cong C_{mn}$ as groups. **Classification of Finite Abelian Groups:** Shown by example: Q: What is #(up to isom.) abelian groups of order 600? A: $600 = (2^3)(3^1)(5^2)$ is prime factorization, and the ans is the prod. of the partition numbers of the powers all w/ each other $(p(3) \cdot p(1) \cdot p(2)) = 6$. $\sigma \in S_n$ an **even** permutation if σ a product of evenly many transpositions, and odd permutation defined similarly. Helpful: partition $p(2) = 2, p(3) = 3, p(4) = 5, p(5) = 7, p(6) = 11$, remember using that these are the first few primes.

Elementary Complex Analysis

$z \in \mathbb{C}$ has $z = x + iy$ for $x, y \in \mathbb{R}$ (cartesian form) and $z = re^{i\theta}$ ($r \in [0,\infty)$, $\theta \in \mathbb{R}$ (polar form). $r = |z| = \sqrt{x^2 + y^2}$, $x = r\cos\theta$, $y = r\sin\theta$. $\theta = \arg(z)$ not unique. $\mathrm{Arg}(z) = \arctan(y/x)$ or $\arctan(y/x) + \pi$ (context-dependent). $z/w = z\overline{w}/|w|^2 = e^{i(\theta_z - \theta_w)} r_z/r_w$. $\mathrm{Log}(z) = \log(|z|) + i\,\mathrm{Arg}(z)$. $z^{x+iy} = z^x (z^y)^i$. $\sin(z) = (e^{iz} - e^{-iz})/2i = -i\sinh(iz)$. $\cos(z) = (e^{iz} + e^{-iz})/2 = \cosh(iz)$. $|\sin|, |\cos|$ unbdd on $\mathbb{C}$. $f: \mathbb{C} \to \mathbb{C}$. f is diff. at z_0 if $f'(z_0) := \lim_{w \to 0}(f(z+w) - f(z))/w$ exists. $f = u + iv$, u, v real fcns. **Cauchy-Riemann Equations:** $u_x = v_y$, $u_y = -v_x$. f diff. at $z_0 \Rightarrow$ C.R. at z_0. C.R. at z_0&u_x,u_y,v_x,v_y cts in open nhd of $z_0 \Rightarrow f$ diff. at z_0. A **domain** is an open connected subset (of e.g. $\mathbb{C}$). Let D be a domain. f is **holomorphic on** domain D if f is diff. at every $z \in D$. f is **holomorphic at** $z_0 \in \mathbb{C}$ if f is holomorphic on some domain containing z_0. *f can be diff. at z_0 but not holomorphic at z_0!* f hol. on $D \Rightarrow$ C.R. eqns hold on D and $f \in C^\infty$. C.R. at $z_0 \in D$, u_x,u_y,v_x,v_y cts on $D \Rightarrow f$ hol. on D. $f: D \to \mathbb{C}$, f **analytic** at $z_0 \in D$ if $\exists$ power series $\Sigma_0^\infty a_n (z - z_0)^n$ converging to f in open nhd U of z_0. f **analytic on** D if f analytic at all $z \in D$. f analytic on $D \Leftrightarrow f$ holomorphic on D. Let R be radius of convergence of $L = \sum a_n (z - z_0)^n$. **Cauchy-Hadamard Theorem:** $1/R = \limsup |a_n|^{1/n}$. L cvgs (i) absolutely in $\{z : |z - z_0| < R\}$, (ii) uniformly in $\{z : |z - z_0| < R - \varepsilon\}$ for any $\varepsilon > 0$. We cannot say anything ab the boundary $|z - z_0| = R$. f is **entire** if f is holomorphic on all of $\mathbb{C}$, e.g. $f = e^z$,cos,sin,polys,etc. $\mathrm{Log}(z): \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ only analytic where defined. f is **meromorphic** if analytic except at *isolated* pts, called **poles**. E.g. $f = e^z/z^2(z+1)$ has poles $z = 0, -1$, is meromorphic. z_0 is a **simple pole** (i.e. pole of order 0) if z_0 is a removable singularity (i.e. if the limit exists, e.g. $\sin(z)/z$). z_0 is a **pole of order** n if n is the smallest s.t. $(z - z_0)^n f(z)$ is analytic at z_0. If $n = \infty$ z_0 is called an **essential singularity** (e.g. the expansion of $e^{1/z}$ shows it has $z_0 = 0$ as an essential singularity). Let f be meromorphic but analytic on an (open) annulus around z_0, $A = \{r < |z - z_0| < R\}$, $r \in [0,\infty)$, $R \in (r, \infty]$. The **Laurent** series of f in an annulus centered at z_0 is $\Sigma_{-\infty}^\infty a_n (z - z_0)^{-n}$. Examples: $1/z$ about $z_0 = 0$ (on any annulus w/o 0) is its own Laurent series. But centered around $z_0 = 1$, say in annulus $A = \{1 < |z - 1| < 2\}$, have (by typical power series manipulations) that $1/z = 1/(1 - (-1)(z-1)) = (z-1)^{-1} - (z-1)^{-2} + (z-1)^{-3} - \cdots$. *The poles outside the annulus do not contribute to* a_{-1}. If asked to find Laurent series of f which has $\geqslant 2$ pole, decompose f w/ partial fractions, do each separately, then sum. If asked to find Laurent series of $2/(z-2)^2$, use that $2/(z-2)^2 = -2\frac{d}{dz} 1/(z-2)$.

Complex Integration

Let $C \subset \mathbb{C}$ be an oriented smooth curve $z(t) = u(t) + iv(t)$, $t \in [a,b]$, define $\int_C f(z)dz = \int_a^b f(z) z'(t) dt$, where $z'(t) = u'(t) + iv'(t)$. i is treated as any constant. **FTOC for Complex Integrals:** If $F' = f$ *in a domain D containing C* then $\int_{C^{z_1 \to z_2}} f(z)dz = F(z_2) - F(z_1)$. E.g. $\int_C iz dz = i\int_C z dz = i(z_2^2/2 - z_1^2/2)$. $\frac{d}{dz}\frac{1}{n+1}(z - z_0)^{n+1} = (z - z_0)^n = 0$ for all $n \in \mathbb{Z} \setminus \{-1\}$ on $\mathbb{C}$ (or $\mathbb{C} \setminus \{z_0\}$ if the power is negative). If $z_0 = x_0 + iy_0$, then $\frac{d}{dz}\mathrm{Log}(z - z_0) = \frac{1}{z - z_0}$ on $\mathbb{C} \setminus (-\infty, x_0] \times \{y_0\}$, but not on all of $\mathbb{C}$. Let C be a closed smooth simple curve orientd ccw. (1) C encloses $z_0 \Rightarrow \oint_C \frac{1}{z - z_0} dz = 2\pi i$. (2) If C does *not* enclose z_0, then $\oint_C \frac{1}{z - z_0} dz = 0$. 3. $\oint_C f(z)dz = 2\pi i a_{-1}$, where a_{-1} is the coefficient of z^{-1} in the Laurent series of f in an annulus around containing C centered at z_0. Define the **residue** of a pole z_0 of f of order k by $\mathrm{Res}(z_0, f) = \frac{1}{(k-1)!}\lim_{z \to z_0}((z - z_0)^k f(z))^{(k-1)}$, and for $k = \infty$ set $\mathrm{Res}(z_0, f) = a_{-1}$, the coef. of z^{-1} in the Laurent series of f in an annulus containing C centered at p_i. **Residue Theorem:** $\oint_C f(z)dz = 2\pi i \sum_{\text{poles in } C} \mathrm{Res}(z_j, f)$. (Also gives an alternate way to calculate $a_{-1} = \sum_{\text{poles in } C} \mathrm{Res}(z_j, f)$.) **Strategy to calculate $\oint_C f(z)dz$:** (1) If possible, break f into summands, compute the integrals for every summand separately, and add the results. (2) Identify poles p_i enclosed by C and their orders. (3) Calculate the residues $\mathrm{Res}(p_i, f)$ (with respect to the Laurent series of an annulus containing C, centered at p_i). (4) Eliminate analytic summands of f. Finding a_n of Laurent Series in an annulus around z_0: $a_0 = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0} dz = f(z_0)$ (since f analytic at all points within and on simple closed path C containing point z_0 and hence power series at z_0, $f(z_0) = \sum a_n (z - z_0)^n$, vanishes everywhere except the term $a_0(z - z_0)^0 = a_0$, so $f(z_0) = a_0$). $a_n = \frac{1}{2\pi i}\oint_C f(z)/(z - z_0)^{n+1} dz$. $a_{-1} = \frac{1}{2\pi i}\oint_C f(z)/(z - z_0)^{-1} dz$. $f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$ (by similar reasoning to the first bullet point, except we multiply by $n!$ to account for the product rule bringing the exponents down to make $n!$). Taking $C = \{z : |z - z_0| = R\}$ and $M = \sup_C |f(z)|$, we obtain $|f^{(n)}(z_0)| \leqslant \frac{n!}{R^n} M$. **Liouville's Theorem:** An entire function which is bounded is constant.

ODE Solutions:

First Order: Separable: $y'(x) = f(x)g(y) \Rightarrow \int \frac{dy}{g(y)} = \int f(x)dx + C$. **Homogeneous:** $y' = f(y/x) \Rightarrow$ sub. $u = y/x$ to make separable. **Linear:** $y' + p(x)y = q(x) \Rightarrow y = e^{-\int^x p}(\int e^{\int^x p} q(x)dx + C)$. **Exact:** $P(x,y)dx + Q(x,y)dy$, $P_y = Q_x \Rightarrow u(x,y) = C$, where $du = u_x dx + u_y dy = Pdx + Qdy$. Solving for u gives an implicit eqn of x and y, as desired. **Non-Exact (Special Case):** $P_y \neq Q_x, (M_y - N_x)/M = h(y)$ (is a fcn of y) $\Rightarrow \mu(y) = e^{\int^y h}$; $P_y \neq Q_x, (M_y - N_x)/(-N) = h(x)$ (is a fcn of x, noting the negative sign on $-N$) $\Rightarrow \mu(x) = e^{\int^x h}$ makes $\mu P dx + \mu Q dy = 0$ exact so that we can solve for u to get an implicit equation of x and y as in the exact case above, as desired. A **homogeneous second-order equation with constant coefficients** is $ay''(x) + by'(x) + cy(x) = f(x) \Rightarrow y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$ if $r_1 \neq r_2$ and $y = (C_1 + C_2 x)e^{r_1 x}$ otherwise.