Classical field theories [very unfinished]

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1. Classical field theories

1.1. **Fields and field equations.** A *field theory* is a pair $(\pi, \mathfrak{F})$ consisting of a smooth fiber bundle $\pi\colon E \to M$ and a collection $\mathfrak{F}$ of *fields*, which are just sections $\phi \in \Gamma(M, E)$. Here E is called the *total space* (AKA configuration space, internal space[1]) and M is called the *spacetime* (AKA parameterization space, external space).

Often a field theory comes with a *field equation*, which is a map $\mathcal{E}\colon \mathfrak{F} \to \mathbb{C}^r$ for some $r > 0$. A field $\phi \in \mathfrak{F}$ is *on-shell* if it satisfies $\mathcal{E}\phi = 0$, and ϕ is *off-shell* otherwise. We write $\mathfrak{F}_{\text{shell}}$ for the collection of on-shell fields $\phi \in \mathfrak{F}$. Note that $\mathfrak{F}_{\text{shell}}$ is also called the *phase space* or *trajectory space*. An *observable* of a field theory $(\pi, \mathcal{L})$ is a smooth map $\mathcal{O}\colon \mathfrak{F}_{\text{shell}} \to X$ and the observations are just the values of $\mathcal{O}$ on M.

1.2. **Principle of least action.** The main tool to study field theories is to use the action principle. The *action principle* (AKA the principle of least action) asserts that there is a smooth function $\mathcal{S}\colon \mathfrak{F} \to \mathbb{R}$, called the *action functional*, such that a field $\phi \in \mathfrak{F}$ is on-shell if and only if ϕ is a critical point of $\mathcal{S}$.

It is usually easier to construct and study a field theory with action functionals rather than directly from its field equations. For example, a diffeomorphism $\Phi\colon \mathfrak{F} \to \mathfrak{F}$ acts naturally on functions on $\mathfrak{F}$ (i.e., on observables of the theory) by pullback, so Φ is a symmetry of the field theory with action functional $\mathcal{S}$ if $\Phi^*\mathcal{S} = \mathcal{S}$. It follows that $\Phi(\mathfrak{F}_{\text{shell}}) = \mathfrak{F}_{\text{shell}}$. Conversely, if the symmetries are known, then the requirement for $\mathcal{S}$ to be invariant heavily restricts the possible action functionals.

1.3. **Lagrangian field theories.** A *Lagrangian field theory* $(\pi, \mathcal{L})$ consists of a smooth fiber bundle a smooth map

$$\mathcal{L}\colon \Gamma(M, J^1(\pi)) \to \Gamma(M, \mathrm{Dens}_M),$$

called the *Lagrangian density*, from the collection of 1-jet prolongations $j^k\phi$ of fields $\phi \in \mathfrak{F}$ into the line of (weight 1) densities on M. Using the canonical isomorphism $J^1(\pi) \cong E \oplus \Omega^1(M; E)$ given by $j^1\sigma \mapsto (\sigma, d\sigma)$, we will henceforth write $(x, \sigma(x), d\sigma(x))$ to mean $(j^1\sigma)_x$, the value at x of the 1-jet prolongation.

From the Lagrangian density, we define the *action* of a field $\phi \in \mathfrak{F}$ by the scalar

$$\mathcal{S}(\phi) \coloneqq \int_M \mathcal{L}(j^1\phi), \tag{1}$$

which in turn defines a map $\mathcal{S}\colon \mathfrak{F} \to \mathbb{R}$ called the *action functional*.[2]

[1] Usually "internal space" refers to the standard fiber of the fiber bundle rather than the total space E.

[2] From here, pp. 91–2: "The precise space of functions upon which the functional (1) is to be extremized will depend on any boundary conditions which may be imposed — e.g., the Dirichlet conditions $u = 0$ on $\partial\Omega$ as well as smoothness requirements. More generally, although this is beyond our scope, one may also impose additional constraints, e.g., holonomic (meaning the fields ϕ are jet prolongations $j^1\phi$), non-holonomic, integral, etc."

The action functional $\mathcal{S}\colon \mathfrak{F} \to \mathbb{R}$ is *smooth* at a field ϕ if for all variations $\Phi\colon \mathbb{R}\times M \to E$, the composite $\mathcal{S}\widehat{\Phi}\colon \mathbb{R} \to \mathfrak{F} \to \mathbb{R}$ is smooth, where $\widehat{\Phi}$ is obtained from Φ by currying. A field $\phi \in \mathfrak{F}$ is a *critical point* of $\mathcal{S}$ if $\mathcal{S}$ is smooth and $\delta\mathcal{S} = 0$.

1.4. **Continuous transformation of a field theory.** Consider a a classical field theory $(\pi, \mathcal{E})$ with set of fields $\mathfrak{F}$. A *continuous transformation* of a field $\phi \in \mathfrak{F}$ consists of smooth maps $\Phi\colon \mathbb{R} \times M \to E$ and the flow $\Psi\colon \mathbb{R} \times M \to M$, which we write as $s \mapsto \Phi_s$ and $\varepsilon \mapsto \Psi_\varepsilon$ respectively, such that Ψ is the flow of a smooth vector field $X \in \mathfrak{X}(M)$, $\Phi_0 = \phi$ and $\Psi_0 = \mathrm{id}_M$. Any such map is called a *variation* and a *flow* respectively. For a fixed point $x \in M$, this induces a two-parameter smooth map $(s,t) \mapsto \Phi_s(\Psi_\varepsilon(x))$ defined on a small open neighborhood of the origin in $\mathbb{R}^2$, so by the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} h(\varepsilon,\varepsilon) = \frac{\partial h}{\partial \varepsilon}\bigg|_{(0,0)} + \frac{\partial h}{\partial s}\bigg|_{(0,0)} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} h(\varepsilon,0) + \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} h(0,s) = \frac{\mathrm{d}\Psi_\varepsilon}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} + \frac{\mathrm{d}\Phi_s}{\mathrm{d}s}\bigg|_{s=0}.$$

Therefore, for the operators

$$\delta \coloneqq \frac{\mathrm{d}\Psi_\varepsilon}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \qquad \text{and} \qquad \mathrm{d} \coloneqq \frac{\mathrm{d}\Phi_s}{\mathrm{d}s}\bigg|_{s=0},$$

the operator $\Delta \coloneqq \delta + \mathrm{d}$ gives the "total variation" under the "infinitesimal transformation" associated to the continuous transformation of our field theory at hand.

The following result describes how fields can vary under transformations of *Lagrangian* field theories.

Lemma 1. *For a Lagrangian field theory with space of fields $\mathfrak{F}$, Lagrangian density $\mathcal{L}$, and action functional $\mathcal{S}$. with space of fields $\mathfrak{F}$, a field $\phi \in \mathfrak{F}$ and the action functional $\mathcal{S}$,*

- $\Delta\phi = \delta\phi + \mathfrak{L}_X\phi$ *where $\mathfrak{L}_X\phi$ is the Lie derivative of ϕ along the flow of the infinitesimal generator X;*
- $\Delta\mathcal{L} = \delta\mathcal{L} + \mathfrak{L}_X\mathcal{L}$*; and*
- $\Delta\mathcal{S} = \int_M \Delta\mathcal{L} = \int_M (\delta\mathcal{L} + \mathfrak{L}_X\mathcal{L})$.

Proof. See here for the details. □

1.5. **Euler–Lagrange equations for Lagrangian field theories.** Recall that, given a continuous transformation of a field theory and a field $\phi \in \mathfrak{F}$, we defined the operators d, δ, and Δ. By noting the behavior of $\mathcal{L}(\phi)$ and $\mathcal{S}(\phi)$ under the infinitesimal continuous transformation $\Delta\phi$, we obtain definitions for $\Delta\mathcal{L}$ and $\Delta\mathcal{S}$. Here we work to better understand the behavior of fields ϕ that are critical points of the action functional on a fixed coordinate chart $U \subseteq M$.

Suppose $\mathcal{L}$ is a Lagrangian density. Fix a precompact subset $K \subseteq M$ with smooth boundary and let $\mathfrak{F}_K$ denote the collection of fields $\phi \in \mathfrak{F}$ that vanish outside K. For simplicity, we assume K is contained in the coordinate chart $U \subseteq M$. Now, for fields $\phi, \eta \in \mathfrak{F}_K$, we can compute the first variation by noting that we can integrate the Lagrangian over K instead of M, as $\mathcal{L}$ is local and if $(x, \phi(x), \mathrm{d}\phi(x))$ is constant for all x outside of K, the variation and thus the integral is zero. We can thus write $\delta\mathcal{S}$ as

$$\delta\mathcal{S} = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \int_K \Phi_s^*\mathcal{L} = \int_K \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \Phi_s^*\mathcal{L} = \int_K \delta\mathcal{L}.$$

Where x^μ are local spacetime coordinates on chart U, we have

$$\mathcal{L}(x, \phi(x), \mathrm{d}\phi(x)) = L(x, \phi(x), \mathrm{d}\phi(x)) \left|\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n\right|$$

for some $L \in C^\infty(J^1(M, E))$. We call L the *Lagrangian* with respect to this local coordinate system. In these coordinates we can use the multivariable chain rule to write

$$\begin{aligned}\delta\mathcal{L} &= \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} L(x, \Phi_s(x), \mathrm{d}\Phi_s(x)) \left|\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n\right| \\ &= \left[\frac{\partial L}{\partial\phi}\cdot\delta\phi + \frac{\partial L}{\partial(\mathrm{d}\phi)}\cdot\mathrm{d}(\delta\phi)\right]_{(x,\phi(x),\mathrm{d}\phi(x))} \left|\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n\right|\end{aligned}$$

where $\cdot$ denotes real scalar multiplication. Thus

$$\delta\mathcal{S} = \int_K \left[\frac{\partial L}{\partial\phi}\cdot\delta\phi + \frac{\partial L}{\partial(\mathrm{d}\phi)}\cdot\mathrm{d}(\delta\phi)\right]_{(x,\phi(x),\mathrm{d}\phi(x))} \left|\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n\right|.$$

Using integration by parts on the second term, namely that $\int \mathrm{d}(uv) = \int u\,\mathrm{d}v + \int v\,\mathrm{d}u$ where $u \coloneqq \frac{\partial L}{\partial(\mathrm{d}\phi)}$ and $v \coloneqq \delta\phi$, we obtain

$$\delta\mathcal{S} = \int_K \left[\left[\frac{\partial L}{\partial\phi} - \mathrm{d}\left(\frac{\partial L}{\partial(\mathrm{d}\phi)}\right)\right]\cdot\delta\phi + \mathrm{d}\left(\frac{\partial L}{\partial(\mathrm{d}\phi)}\delta\phi\right)\right]_{(x,\phi(x),\mathrm{d}\phi(x))} \left|\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n\right|. \quad (1)$$

By our hypotheses on K we can invoke Stokes' theorem to find that the integral of the second term is equal to

$$\int_{\partial K} \iota_{\partial K}\left[\frac{\partial L}{\partial(\mathrm{d}\phi)}\cdot\delta\phi\right]_{(x,\phi(x),\mathrm{d}\phi(x))} \left|\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n\right|,$$

which is zero because $\delta\phi = 0$ on ∂K. Thus (1) reduces to

$$\delta\mathcal{S} = \int_K \left[\frac{\partial L}{\partial\phi} - \mathrm{d}\left(\frac{\partial L}{\partial(\mathrm{d}\phi)}\right)\right]_{(x,\phi(x),\mathrm{d}\phi(x))}\cdot\delta\phi \left|\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n\right|. \quad (2)$$

Therefore, by the fundamental lemma of the calculus of variations,

$$\delta\mathcal{S} = 0 \qquad \iff \qquad \frac{\partial L}{\partial\phi} - \mathrm{d}\left(\frac{\partial L}{\partial(\mathrm{d}\phi)}\right) = 0$$

The following theorem summarizes our discussion.

Theorem 3 (Euler–Lagrange)**.** *A fields $\phi \in \mathfrak{F}$ is a critical point for the action functional of* *a Lagrangian field theory* $(\pi, \mathcal{L})$ *if and only if* ϕ *satisfies the* Euler–Lagrange equation *for* $\mathcal{L}$*,*

$$(\mathrm{E}\mathcal{L})(\phi) \coloneqq \frac{\partial L}{\partial \phi} - \mathrm{d}\left(\frac{\partial L}{\partial(\mathrm{d}\phi)}\right) = 0.$$

Corollary 4. *A Lagrangian field theory* $(\pi, \mathfrak{F}, \mathcal{L})$ *is a field theory* $(\pi, \mathfrak{F})$ *whose field equations are the Euler–Lagrange equations.*

2. Symmetries and Noether's theorems

A *transformation* of a field theory $(\pi, \mathfrak{F})$ is a map $\mathfrak{F} \to \mathfrak{F}$.

2.1. **Types of symmetries.** Here we follow here. Here we work with a fixed Lagrangian field theory $(\pi, \mathfrak{F}, \mathcal{L})$. A *symmetry* is any transformation Ξ of $\mathfrak{F}$ that preserves the action functional in the sense that

$$\mathcal{S}(\Xi(\phi)) = \mathcal{S}(\phi) \tag{1}$$

for all fields $\phi \in \mathfrak{F}$. If Equation (1) merely holds for fields $\phi \in \mathfrak{F}_{\text{shell}}$, then we call Ξ an *on-shell symmetry.*

2.1.1. *Internal and external symmetries.* An *internal* (resp. *external) symmetry* is a symmetry induced by a transformation of the internal space E preserving M (resp. of the external space M preserving E).

2.1.2. *Continuous and discrete symmetries.* A symmetry is *continuous* if there is a homotopy from identity, that is, if there is a homotopy between the identity transformation and the given transformation. A non-continuous symmetry is called *discrete.*

From here: For continuous symmetries, i.e., elements g of a Lie group G, we are interested in invariance under infinitesimal transformations. By "infinitesimal translations" we mean elements of the Lie algebra $\mathfrak{g}$ of G: the infinitesimal symmetry corresponding to g is the element $\theta = \dot{\gamma}(0)$ where γ is a one-parameter subgroup of G (i.e., a smooth group homomorphism $\gamma\colon \mathbb{R} \to G$) such that $\gamma(0) = g$.

2.1.3. *Local (gauge) and global symmetries.* A symmetry is *global* if it is independent of spacetime, that is, if the same transformation is applied to each point in spacetime. A symmetry is *local*[3] (AKA a global gauge symmetry) otherwise.

The following facts are immediate from the definitions and make the classification of symmetries much easier. Local symmetries (AKA gauge symmetries), discrete symmetries, and internal symmetries are examples of off-shell symmetries.

2.1.4. *Examples of symmetries.* (From here.) We next list the remaining types of symmetry along with examples.

[3] If you're a mathematician, this terminology likely isn't what you expected. After reading the definition, you'll probably think it should be called "pointwise" symmetry. However, the two turn out to be equivalent in the context of smooth bundles.

- **Discrete external symmetry:** Parity (P), time-reversal (T), rotations by $2\pi/N$ degrees.
- **Discrete internal symmetry:** Charge conjugation (C), R-parity in supersymmetric theories, $\mathbb{Z}/n\mathbb{Z}$-symmetry that remains from the color or flavor symmetries in QCD-like theories (e.g., in confined or chiral-symmetry broken phases).
- **Continuous internal symmetry:** Flavor symmetry in quark or lepton sectors. Diffeomorphisms in generally covariant theories, super-diffeomorphisms in supergravities.
- **Continuous external on-shell symmetry:** Spacetime translations, rotations, Lorentz symmetry (or isometries of a given spacetime), conformal symmetry, supersymmetry.

2.1.5. *Other types of symmetry.* A symmetry is a *divergence symmetry* if it transforms the Lagrangian $\mathcal{L}$ as $\mathcal{L} \mapsto \mathcal{L} + \operatorname{div} K$ for some $(n-1)$-form K.[4]

2.2. **Noether's first theorem.** Given that all four fundamental forces can be represented as gauge theories, it essentially states that continuous global symmetries are precisely physically realized symmetries in nature.

A *current* on an n-dimensional smooth manifold is an $(n-1)$-form $j \in \Omega^{n-1}(M)$. We say a current j is *conserved* if the n-form $\mathrm{d}j$ is zero, i.e., if j is closed. Noether's first theorem (AKA Noether's theorem) asserts that for every global symmetry of a Lagrangian field theory $(\pi, \mathfrak{F}, \mathcal{L})$, that is, every global symmetry $\Xi\colon \mathfrak{F} \to \mathfrak{F}$ under which the action functional $\mathcal{S}\colon \mathfrak{F} \to \mathbb{R}$ is invariant, there is a conserved current j_Ξ.

Theorem 1 (Noether's first theorem)**.** *For a Lagrangian field theory $(\pi, \mathcal{L})$, there is a bijective correspondence between continuous global symmetries and on-shell conservation laws.*

More precisely, if $\mathfrak{F}$ denotes the set of fields of $(\pi, \mathcal{L})$, then this correspondence assigns to a continuous global symmetry $\Xi\colon \mathfrak{F} \to \mathfrak{F}$, written infinitesimally as $\phi \mapsto \phi + \Delta\phi$, the current

$$j \coloneqq \frac{\partial L}{\partial(\mathrm{d}\phi)} \delta\phi,$$

called the Noether current*.*

Proof. Suppose we have a continuous global symmetry, i.e., that $\Delta\mathcal{S} = 0$. On a coordinate chart $U \subseteq M$ with local coordinates x^μ, we can rewrite (1) for on-shell ϕ as

$$\delta\mathcal{S} = \int_K \left[\mathrm{d}\left(\frac{\partial L}{\partial(\mathrm{d}\phi)}\delta\phi\right)\right]_{(x,\phi(x),\mathrm{d}\phi(x))} \left|\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n\right|.$$

[4] Some call transformations preserving the Lagrangian density (as opposed to the action functional) a symmetry and prefer to call what we are calling a divergence symmetry a *quasisymmetry*. In this language, Noether's first theorem asserts that global quasisymmetries are in one-to-one correspondence with conserved currents.

On the other hand, by Cartan's magic formula, we have

$$\mathrm{d}\mathcal{S} = \int_K \mathfrak{L}_X \mathcal{L}(\phi) = \int_K \left[\iota_X \mathrm{d}\mathcal{L} + \mathrm{d}\iota_X \mathcal{L}\right](\phi) = \int_K \mathrm{d}\iota_X \mathcal{L}(\phi) = \int_K \mathrm{div}(X)\mathcal{L}(\phi).$$

where we used $\mathrm{d}\mathcal{L} = 0$ for the third equality (by the Poincaré lemma, since $\mathcal{L}$ is a top form) and this for the last equality. Thus, in local coordinates,

$$\Delta\mathcal{S} = \delta\mathcal{S} + \mathrm{d}\mathcal{S} = \int_K \left[\mathrm{d}\left(\frac{\partial L}{\partial(\mathrm{d}\phi)}\delta\phi\right) + \mathrm{div}(X)L\right]_{(x,\phi(x),\mathrm{d}\phi(x))} \left|\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n\right| = \int_K \mathrm{d}j$$

where

$$j = \frac{\partial L}{\partial(\mathrm{d}\phi)} - \mathrm{div}(X)L.$$

Since $\Delta\mathcal{S} = 0$,

$$j = \Delta\mathcal{S} = \int_K \left[\delta\mathcal{S} + \mathrm{d}\mathcal{S}\right](\phi) = \int_K \mathrm{d}\left[\frac{\partial L}{\partial(\mathrm{d}\phi)}\delta\phi + \iota_X\mathcal{L}(\phi)\right]_{(x,\phi(x),\mathrm{d}\phi(x))}.$$

Since $\Delta\mathcal{S} = 0$, this integral must vanish for any K, which implies $\mathrm{d}j = 0$. Thus, j is conserved on-shell, completing the proof. □

2.3. **Noether's second theorem.**

Theorem 1 (Noether's second theorem)**.** *For a Lagrangian field theory $(\pi, \mathcal{L})$ with set of fields $\mathfrak{F}$, if a local divergence symmetry $\phi \mapsto \delta\phi$ transforms the Lagrangian density as $\mathcal{L} \mapsto \mathcal{L} + \mathrm{div}\, K$ for some $(n-1)$-form $K \in \Omega^{n-1}(J^1(M,E))$, then we have the* Noether identity

$$\mathrm{d}(J - K) = 0,$$

where again J is the Noether current.

Proof. Exercise. Use the Euler–Lagrange equations and the Bianchi identity. □

3. Gauge theories

Gauge theory is a language of studying physical systems, usually in the context of (possibly quantum) Lagrangian field theories, by studying their symmetries. More precisely, a *G-gauged field theory* is a field theory $(\pi, \mathfrak{F})$ whose field equations are invariant under bundle automorphisms of the G-frame bundle of π.

The fundamental principle of gauge theory is that fields are sections of principal G-bundles $P \to M$ and that the laws of physics are differential equations (Euler—Lagrange equations) that are gauge-invariant, i.e., invariant under the right action of G (called the symmetry group or the gauge group) in the sense that if a section $s \in \Gamma(M,P)$ is a solution, then so is $s \triangleleft g$ for all $g \in G$. We first need some terminology.

Let G be a Lie group. A field theory $(\pi, \mathfrak{F})$ is *G-gauged* if the structure group of π can be reduced to G. A *gauge transformation* is a G-bundle automorphism $P \to P$, and the group $\mathfrak{G}(P)$ of gauge transformations is called the *gauge group*.

Warning 1. In the mathematical literature, the group $\mathfrak{G}(P)$ of principal bundle automorphisms of $P \to M$ is called the gauge group, whereas in the physics literature the structure group G is usually called the gauge group. These are not the same in general; indeed, $\mathfrak{G}(P)$ is usually an infinite-dimensional manifold.

In practice, we think of $P \xrightarrow{\pi} M$ as a frame bundle of the associated bundle $P \times_G V$ for some faithful r-dimensional G-representation $\rho\colon G \to \mathrm{GL}(V)$, so that G acts by change of basis as a subgroup of $\mathrm{GL}(V)$ and $P \times_\rho V$ has structure group G. A section of $P \times_\rho V$ is called a *matter field* (AKA particle field).

Note that any gauge-related terminology regarding vector bundles should be understood to mean for their frame bundles, which of course are principal bundles.

3.1. **Choosing a gauge.** A *(local) gauge* on P consists of a choice of local trivialization of each coordinate chart of P. The following result says that we can view these as pullbacks of principal sections.

Proposition 1. *A choice of gauge on P is equivalent to a choice of principal section $\sigma \in \Gamma(U, P)$ on each coordinate chart $U \subseteq M$.*

Proof. For a local section $\sigma \in \Gamma(U, P)$, there is a canonical right G-equivariant isomorphism $\phi_U\colon U \times G \to \pi^{-1}(U)$ given by $(x, g) \mapsto \sigma(x) \triangleleft g$, which is equivariant for the right G-action because $(x, g) \triangleleft h = (x, gh) \mapsto \sigma(x) \triangleleft (gh) = (\sigma(x) \triangleleft g) \triangleleft h$, and injective (resp. surjective) because the right G-action on the fibers is free (resp. transitive), and hence is a local trivialization of P over U.

Conversely, given a local trivialization of P over U, i.e., a right G-equivariant isomorphism $\phi_U\colon U \times G \to \pi^{-1}(U)$, there is a canonical section $\phi_U^0 \in \Gamma(U, P)$, called the *identity section* with respect to this local trivialization, given by $\phi_U^0(x) \coloneqq \phi_U(x, 1)$. □

3.2. **Gauge transformations.** A *gauge transformation* (AKA local gauge transformation, gauge transformation of the second kind, gauge symmetry) is a principal bundle automorphism $\Phi\colon P \to P$, i.e., a right G-equivariant homeomorphism $P \to P$ of the form $(x, g) \mapsto (x, \phi_U(x, g))$ for some map $\phi_U\colon U \times G \to G$.

Proposition 1. *A gauge transformation $\Phi\colon P \to P$ of a principal G-bundle P over $U \subseteq M$ is equivalent to a map $\gamma\colon P \to G$ satisfying*

$$\gamma(p \triangleleft g) = g^{-1}\gamma(p)g \qquad \forall\, p \in P, g \in G. \tag{2}$$

Moreover, where σ and σ' are the identity sections corresponding to the local trivializations in the source and target of Φ respectively, we have $\sigma'(x) = \sigma(x) \triangleleft \gamma(p)$.

Proof. Let $\Phi\colon P \to P$ be a gauge transformation. Since Φ is a bundle map, it sends fibers to fibers, meaning $\pi\Phi(p) = \pi(p)$. Thus, if $\pi(p) = x$, $\Phi(p) \in P_x$, then $\pi(\Phi(p)) = p$, i.e., p and $\Phi(p)$ lie in the same fiber. Since the right G-action is free and transitive on P, we can write $\Phi(p) = p \triangleleft \gamma(p)$ for some $\gamma(p) \in G$. This gives a map $\gamma\colon P \to G$. We can now rephrase the right G-equivariance of Φ as follows.

$$\begin{aligned}\Phi(p) \triangleleft g = \Phi(p \triangleleft g) &\iff (p \triangleleft \gamma(p)) \triangleleft g = (p \triangleleft g) \triangleleft \gamma(p \triangleleft g)\\ &\iff p \triangleleft (\gamma(p)g) = p \triangleleft (g\gamma(p \triangleleft g))\\ &\iff \gamma(p)g = g\gamma(p \triangleleft g) \iff \gamma(p \triangleleft g) = g^{-1}\gamma(p)g.\end{aligned}$$

Conversely, given a map $\gamma\colon P \to G$ satisfying $\gamma(p \triangleleft g) = g^{-1}\gamma(p)g$, one can check the map $\Phi\colon P \to P$ given by $\Phi(p) \coloneqq p \cdot \gamma(p)$ is a bundle isomorphism. This proves the first assertion.

For the second assertion, note that the identity sections σ and σ' corresponding to the local trivializations respectively before and after the gauge transformation Φ satisfy $\sigma'(x) = \sigma(x) \triangleleft \gamma(\sigma(x))$, which can be seen by applying Φ to $\sigma(x)$ and then using the definition of γ as the gauge transformation's action on the fibers. □

A choice of gauge (local section $\phi^0_U\colon U \to P$) in a principal G-bundle P induces a local trivialization $\psi\colon U \times V \to P \times_G V$ over U in any associated vector bundle via $\psi(x, v) = [\phi^0_U(x), v]$. Conversely, every local trivialization of $P \times_G V$ arises from such a gauge in P. Furthermore, a gauge transformation $\Phi\colon P|_U \to P|_U$ defined by a map $\gamma\colon U \to G$ transforms ψ into a new trivialization $\psi'(x, v) = [\phi^0_U(x) \cdot \gamma(x), v]$ via the representation $\rho\colon G \to \mathrm{GL}(V)$.

3.3. **Gauge fields.** A *gauge field* (AKA *gauge potential* is a principal connection 1-form $\omega \in \Omega^1(P; \mathfrak{g})$. Given a gauge field ω, a choice of local gauge $\sigma\colon U \to P|_U$, induces by pullback a $\mathfrak{g}$-valued 1-form $A \coloneqq \sigma^*\omega \in \Omega^1(M; \mathfrak{g})$, called a *local gauge potential* (AKA local potential, local vector potential). For local coordinates x^μ on U and induced local frame ∂_μ for TM, we write A_μ for the $\mathfrak{g}$-valued function $A(\partial_\mu)$ on U.

From here: "In the case of local, dynamical symmetries, associated with every charge is a gauge field; when quantized, the gauge field becomes a gauge boson. The charges of the theory "radiate" the gauge field. Thus, for example, the gauge field of electromagnetism is the electromagnetic field; and the gauge boson is the photon."

We now study the behavior of the principal connection 1-form ω and its the local gauge potential A corresponding to a given gauge. A local gauge transformation $\Phi\colon P|_U \to P|_U$ gives by pushforward another principal connection 1-form $\Psi \cdot \omega \coloneqq \Psi_*\omega$, so that $(\Psi \cdot \omega)_{\phi(p)}((\mathrm{d}\Phi)v) = \omega_p(v)$. If G is a Lie group, one can show (see here, p. 33–4)

$$(\Psi \cdot A) = \Psi_U \cdot A \cdot \Psi_U^{-1} - (\mathrm{d}\Psi_U) \cdot \Psi_U^{-1} \tag{1}$$

where $\Psi_U\colon U \to G$ is the map with $\Psi(x, g) = (x, \Psi_U(x, g))$. (There is a similar formula involving the Maurer–Cartan form if G is not a Lie group).

3.4. **Gauge field strength.** Recalling that the curvature Ω of ω is given by the formula $\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega]$

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Consider the (local) gauge transformation corresponding to an identity section $\phi_U^0 \colon U \to \mathrm{GL}_n(\mathbb{C})$ with respect to a fixed gauge on U. By considering the composition $\check{\gamma} := \rho\gamma \colon U \to \mathrm{GL}(V)$ and choosing a local gauge on the associated vector bundle $P \times_\rho V$, we obtain a matrix field (i.e., a $(1,1)$-tensor field) $\gamma^\beta{}_\alpha \colon U \to \mathrm{GL}_n(\mathbb{C})$. With respect to this local gauge of the associated bundle, a matter field Φ has local expression $\vec{\Phi}$, and transforms under this change of coordinates by $\Phi \mapsto \Phi'$ where $\Phi'^\beta = (\gamma^{-1})^\beta{}_\alpha \Phi^\alpha$, where γ^{-1} is the inverse matrix to γ in this basis.

This means G acts on the fibers $(P \times_\rho V)_x$ of the associated bundle by $g \in G$ sending $[p, v] \in (P \times_\rho V)_x$ to $[p, \rho(g)^{-1} v] \in (P \times_\rho V)_x$, which is a *right* G-action, and we can simply write $v \triangleleft g = \rho(g)^{-1} \triangleright v$. In other words, for a principal G-bundle P and a G-representation V, G acts on the fibers of the associated bundle on the right by change of basis dictated by the change in local trivialization of P.

3.5. **Intertwiners and conservation laws.** The following is adapted from here, pp. 8–9. Any physical process caused by a force is described by an intertwining operator for the G-action, that is, a G-equivariant linear operator. Specifically, if V and W are finite-dimensional Hilbert spaces on which a group G acts unitarily, then an operator $F \colon V \to W$ is *intertwining* if

$$F(g\psi) = gF(\psi) \qquad \forall\, \psi \in V, g \in G.$$

In quantum mechanics, symmetries yield conserved quantities: If G is a Lie group with a unitary representation on V and W, they become representations of $\mathfrak{g}$, the Lie algebra of G, and any intertwining operator F satisfies

$$F(T\psi) = TF(\psi) \qquad \forall\, T \in \mathfrak{g}, \psi \in V.$$

If $\psi \in V$ is an eigenvector of T with eigenvalue $i\lambda$, then

$$TF(\psi) = i\lambda F(\psi),$$

so λ is "conserved" by F.

Each element $T \in \mathfrak{g}$ acts as a skew-adjoint operator on any unitary representation of G. Recall that in quantum mechanics, physicists prefer self-adjoint operators, as they have real eigenvalues. In quantum mechanics, self-adjoint operators are known as "observables" as they admit real eigenvalues, which are the "observations" or "measurements". To obtain an observable, divide T by i, resulting in T/i, which is self-adjoint.

Mathematics	Physics
Principal bundle	Instanton sector or charge sector
Structure group	Gauge group or local gauge group
Gauge group	Group of global gauge transformations or global gauge group
Gauge transformation	Gauge transformation or gauge symmetry
Change of local trivialization	Local gauge transformation
Local trivialization	Gauge
Choice of local trivialization	Fixing a gauge
Functional defined on the space of connections	Lagrangian of gauge theory
Object does not change under the effects of a gauge transformation	Gauge invariance
Gauge transformations that are covariantly constant with respect to the connection	Global gauge symmetry
Gauge transformations that are not covariantly constant with respect to the connection	Local gauge symmetry
Connection	Gauge field or gauge potential
Curvature	Gauge field strength or field strength
Induced connection/covariant derivative on associated bundle	Minimal coupling
Section of associated vector bundle	Matter field
Term in Lagrangian functional involving multiple different quantities (e.g., the covariant derivative applied to a section of an associated bundle, or a multiplication of two terms)	Interaction
Section of real or complex (usually trivial) line bundle	(Real or complex) Scalar field

Table 1: Comparison of concepts in mathematical and physical gauge theory. Adapted from here.