Canonical quantization.[1] For a smooth manifold M (AKA the *configuration space*), velocity vectors live in TM, and momentum vectors live in $T^*M = \Omega^1 M$ (AKA the *phase space*). In other words, velocity vectors are tangent vectors $v = (v^i)$ and momentum "vectors" are really covectors (1-forms) $p = (p_i)$. Then the *state* of a classical system is given by a point in the phase space. The Euler–Lagrange equations are

$$\frac{\partial L}{\partial x^i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}^i},$$

which comprises n second-order PDEs, and whose solutions are the equations of motion for the state (p, q) in phase space. To get the Hamiltonian H of a (classical) Lagrangian field theory $(M, L, \mathcal{F})$, define the *momentum* p_i conjugate to the position x^i by

$$p_i \coloneqq \frac{\partial L}{\partial x^i}.$$

If we are able to solve for velocity $\dot{x}$ in terms of the position x and momentum p, (which we cannot do for general relativity and thus is why quantizing this theory is hard), then we can define the *Hamiltonian* H by

$$H(x, p) \coloneqq p_i x^i - L(q, \dot{q}). \tag{$*$}$$

Now compute $\mathrm{d}H$ twice, first by applying the exterior derivative d to Equation ($*$), and then again by the chain rule. The resulting equations are called *Hamilton's equations*:

$$\begin{cases} \dot{x}^i = \frac{\partial H}{\partial p_i}, \\ \dot{p}_i = -\frac{\partial H}{\partial x_i}. \end{cases}$$

A great way of thinking about this stuff is to define the *Poisson bracket* for *obserbables* (which, as we are still in the classical setting, are simply functions defined on the phase space $\Omega^1(M)$), which on the algebra of observables is a Lie bracket given by

$$\{f, g\} \coloneqq \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i} - \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p_i},$$

using of course the index summation convention as usual. Moreover, we have the Leibniz law $\{f, gh\} = \{f, g\}h + g\{f, h\}$. This is useful because Hamilton's equations are just

$$\begin{cases} \dot{x}^i = \{H, x^i\}, \\ p_i = \{H, p_i\}. \end{cases}$$

More generally, if f is any observable (think: "field", i.e., function on phase space $\Omega^1(M)$), then by the multivariable chain rule

$$\dot{f} = \frac{\mathrm{d}}{\mathrm{d}t} f(p, q) = \frac{\partial f}{\partial x^i}\dot{x}^i + \frac{\partial f}{\partial p_i}\dot{p}_i = \{H, f\},$$

and thus *the rate of change of an observable is determined by its Poisson (Lie) bracket with the Hamiltonian.* We thus say the Hamiltonian H *generates* time evolution. Now define *first* *quantization*

$$\{f, g\} = k \quad \rightsquigarrow \quad [\widehat{f}, \widehat{g}] = -i\hbar\widehat{k}$$

(The factor of i is required for $\widehat{k}$ to be self-adjoint.) When this can be done to *all* observables in a consistent way (which is not always possible), time evolution is now given by

$$\widehat{f}_t = e^{i\widehat{H}t/\hbar}\widehat{f}e^{-i\widehat{H}t/\hbar},$$

so that, in analogy with classical mechanics,

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{f}_t = i[\widehat{H}, \widehat{f}_t],$$

and we again say that the time evolution is *generated* by the Hamiltonian $\widehat{H}$.