TERSE NOTES ON ALGEBRAIC TOPOLOGY

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Conventions

All categories are assumed locally small. We assume all (co)limits are taken over functors defined on small categories. We use "map" to mean morphism in whichever category makes sense the most in the given context, and this will mostly mean continuous maps. We write I for the pointed interval $([0,1],1)$ and identify spaces $X \in \mathcal{T}$ with $X \times \{0\}$. We write Top and Top_* for the categories of topological spaces and pointed topological spaces respectively and continuous maps.

Contents

TODO: Organize sections into sections, subections, subsubsections, possibly even paragraphs and subparagraphs. Also maybe revisit separation into chapters; is it still a good idea?

Chapter 1

Elementary homotopy theory

1.1. Spaces and homotopy

1.1.1. **(Pointed) topological spaces, pairs, and triples.** **TODO: Under construction**

1.1.2. **Wedge sum, smash product, and half-smash products.** Fix $X, Y \in \mathsf{Top}_*$. Define their *wedge sum* $X \vee Y$, *smash product* $X \wedge Y$, and *half-smash product* $X \rtimes Y$ to be $X \vee Y \coloneqq (X \amalg Y)/\{*_X, *_Y\}$, $X \wedge Y \coloneqq (X \times Y)/(X \vee Y)$, and $X \rtimes Y \coloneqq (X \times Y)/(*_X \times Y)$ respectively. In Top_*, there are natural pointed homeomorphisms $X \wedge Y \cong Y \wedge X$ and $X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z$. This correctly suggests $\wedge$ makes Top_* into a symmetric monoidal category.

1.1.3. **CGWH spaces.** A topological space is *compactly generated* (CG) if its open subsets are precisely the open subsets in its compact subspaces. Quotients of CG spaces, locally compact spaces, and first-countable spaces are CG. A topological space is *weak Hausdorff* (WH) if continuous images of compact Hausdorff spaces into the space are closed. Hausdorff (T_2) spaces are WH and WH spaces are T_1.

Let $\mathcal{T}_\circ$ denote the category of compactly generated weak Hausdorff (CGWH) spaces and continuous maps. Examples of CGWH spaces include locally compact Hausdorff (LCH) spaces. The inclusion $\mathcal{T}_\circ \hookrightarrow \mathsf{Top}$ is not an equivalence, since for instance $\prod_{\alpha \in I} \mathbb{R}$ is not in the essential image when I is uncountable. Let $\mathcal{T}_*$ denote the category of pointed CGWH spaces and pointed maps. Many arguments apply in both $\mathcal{T}_\circ$ and $\mathcal{T}_*$; in this case we use $\mathcal{T}$ to mean both.

(Cartesian) products of WH spaces are WH, but this fails for CG spaces. Thus, for general $X, Y \in \mathcal{T}$, the product $X \times Y$ in $\mathcal{T}$ is not the product in Top. While their underlying sets coincide, the topology on the former is obtained by strengthening the usual product topology by adding the subsets that are open in all compact subspaces of the topological product space. More generally, if $X \in \mathsf{Top}$ is not CG, define its k-*ification* kX to be the space sharing an underlying set with X and whose topology is obtained in this way. The story is similar for mapping spaces: the internal hom in Top is the set of maps $X \to Y$ in the compact-open topology. Unfortunately this space also need not be CG, so again we instead study its k-ification $\mathcal{T}_\circ(- \to -)$. The situation is similar in $\mathcal{T}_*$, wherein we denote by $[-,-]_*$ the internal hom, which is the k-ification of the usual pointed mapping space in Top_* with basepoint the constant map.

The upshot of this discussion is that $\mathcal{T}$ is a nice category in which to do algebraic topology.

Theorem 1.1.3.1 ([Str11, Theorems 3.28 and 3.47, adapted])**.** *Let $\otimes$ denote $\times$ when $\mathcal{T} = \mathcal{T}_\circ$ and $\wedge$ when $\mathcal{T} = \mathcal{T}_*$.*

(i) *For all $X, Y, Z \in \mathcal{T}$, the* evaluation map $\mathrm{ev}\colon \mathcal{T}(X \to Y) \otimes X \to Y$ *and* composition map $\circ\colon \mathcal{T}(X \to Y) \otimes \mathcal{T}(Y \to Z) \to \mathcal{T}(X \to Z)$ *are continuous, and the canonical* currying map $\alpha\colon \mathcal{T}(X \otimes Y \to Z) \to \mathcal{T}(X \to \mathcal{T}(Y,Z))$ *is a homeomorphism that is is natural in each of $X, Y, Z \in \mathcal{T}$.*

(ii) *Every diagram of spaces in $\mathcal{T}$ has a limit and colimit in $\mathcal{T}$. Furthermore, the forgetful functors $\mathcal{T}_* \to \mathcal{T}_\circ$ and $\mathcal{T}_\circ \to \mathsf{Set}$ respect limits.*

1.1.4. **(Un)pointed adjunction. Suspension-loop space adjunction.** For $X \in \mathcal{T}_*$ and a map $f\colon X \to Y$ in $\mathcal{T}_\circ$, set $X_+ := (X \amalg \{*\}, *) \in \mathcal{T}_*$ and define $f_+\colon X_+ \to Y_+$ be $f_+|_X = f$ by $f(*) = *$. For $X = (X, *) \in \mathcal{T}_*$ and $f\colon X \to Y$ in $\mathcal{T}_*$, set $X_-\colon X := X \in \mathcal{T}_\circ$ and define $f_-\colon X_- \to Y_-$ by $f_-(x) := f(x)$. Then we have an adjunction $(-)_+ \dashv (-)_-$, since for $X \in \mathcal{T}_\circ$ and $Y \in \mathcal{T}_*$ we have $\mathcal{T}_*(X_+ \to Y) \cong \mathcal{T}_\circ(X \to Y_-)$. Indeed, a pointed map $X_+ \to Y$ is an unpointed map $f\colon X \amalg \{*\} \to Y$ and its restriction to X gives an unpointed map $X \to Y$, while any unpointed map $X \to Y_-$ can be viewed as a pointed map $X \amalg \{*\} \to Y$ by declaring $f(*) := *$.

For $X \in \mathcal{T}_*$, define the *reduced suspension* and *loop space* to be $\Sigma X := S^1 \wedge X$ and $\Omega X := \mathcal{T}_*(S^1 \to X)$ respectively. One can check Σ and Ω are functors $\mathcal{T}_* \to \mathcal{T}_*$. By Theorem 1.1.3.1(i), we have an adjunction $\Sigma \dashv \Omega$.

1.1.5. **Cylinder and path space.** For $X \in \mathcal{T}_\circ$ and $f \in \mathcal{T}_\circ(X \to Y)$ (resp. $X \in \mathcal{T}_*$ and $f \in \mathcal{T}_*(X \to Y)$), define the *standard cylinder* over X to be $\mathrm{Cyl}_\circ(X) := X \times I$ and let $\mathrm{Cyl}_\circ(f) := f \times \mathrm{id}_I$ (resp. $\mathrm{Cyl}_*(X) := X \rtimes I$ and let $\mathrm{Cyl}_*(f) := f \rtimes \mathrm{id}_I$), often simply denoted $\mathrm{Cyl}(X)$ and $\mathrm{Cyl}(f)$, and define the *standard path space* of X to be $\mathrm{Path}_\circ(X) := \mathcal{T}_\circ(I \to X)$ (resp. $\mathrm{Path}_*(X) := \mathcal{T}_*(I \to X)$), often simply denoted $\mathrm{Path}(X)$.

1.1.6. **The homotopy relation.** For maps $f, g\colon X \to Y$ in $\mathcal{T}$, a *homotopy* from f to g is a map $H\colon \mathrm{Cyl}(X) \to Y$ such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, hook, "\iota_0"] \arrow[dr, "f"'] & \mathrm{Cyl}(X) \arrow[d, "H"] & X \arrow[l, hook', "\iota_1"'] \arrow[dl, "g"] \\
& Y &
\end{tikzcd}$$

We write $H\colon f \simeq g$ if H is a homotopy from f to g. Equivalently, by using the currying natural homomorphism $\alpha\colon \mathrm{Map}(X \times I \to Y) \xrightarrow{\cong} \mathrm{Map}(I \to \mathrm{Map}(X, Y))$, a homotopy from f to g is a map $\omega\colon I \to [X, Y]_{\mathcal{T}}$ with $\omega(0) = f$ and $\omega(1) = g$.

The *homotopy category* $\mathsf{Ho}\,\mathcal{T}$ of $\mathcal{T}$ has objects of $\mathcal{T}$ and morphisms homotopy classes of maps, meaning the space of maps quotiented by the homotopy relation. This makes $\mathsf{Ho}\colon \mathcal{T} \to \mathsf{Ho}\,\mathcal{T}$ a functor given by $X \mapsto X$ and $f \mapsto [f]$, and in fact Ho is the universal homotopy preserving functor in the sense that and any homotopy-preserving functor $F\colon \mathcal{T} \to \mathcal{C}$ factors uniquely through $\mathsf{Ho}\colon \mathsf{Ho}\,\mathcal{T} \to \mathcal{C}$.

Spaces $X, Y \in \mathcal{T}$ are *homotopy equivalent* in $\mathcal{T}$, written $X \simeq Y$, if $X \cong Y$ in $\mathsf{Ho}\,\mathcal{T}$. Examples of homotopy equivalences include any *deformation retraction (DR)* of a space X onto a subspace A, i.e., a homotopy $H_\bullet\colon X \times I \to A$ with $H_t|_A = \mathrm{id}_A$ is a homotopy equivalence with homotopy inverse the inclusion $A \hookrightarrow X$.

The *homotopy class* of X is the equivalence class of X under the homotopy relation. We write $\langle X, Y\rangle$ and $[X, Y]$ for $(\mathsf{Ho}\,\mathcal{T}_\circ)(X \to Y)$ and $(\mathsf{Ho}\,\mathcal{T}_*)(X \to Y)$ respectively. As $\wedge$ and $\mathcal{T}_*(- \to -)$ are homotopy preserving in both slots, Ho maps the currying maps to homeomorphisms $[X \wedge Y, Z] \cong [X, \mathcal{T}_*(Y \to Z)]$, natural in each of $X, Y, Z \in \mathcal{T}_*$. Thus for $X, Y \in \mathcal{T}$ with $X \simeq Y$, we have $\Sigma X \simeq \Sigma Y$ and $\Omega X \simeq \Omega Y$.

1.1.7. **Contractible spaces and nullhomotopic maps.** We call $X \in \mathcal{T}$ *contractible* if $X \simeq *$. For example, $\mathbb{R}^n$ is contractible because the map $H_\bullet\colon \mathbb{R} \times I \to \mathbb{R}$ given by $H_t(x) = x(1-t)$ is an equivalence of $\mathbb{R}^n$ and the basepoint 0 with homotopy inverse the inclusion map $0 \hookrightarrow \mathbb{R}^n$. For $Y \in \mathcal{T}$, $Y \simeq * \iff \mathrm{id}_Y \simeq * \iff [X, Y] = *$ for all $X \in \mathcal{T} \iff [Y, Z] = *$ for all $Z \in \mathcal{T} \iff$ the projection $W \times Y \twoheadrightarrow W$ is a homotopy equivalence.

We call $f \in \mathcal{T}(X \to Y)$ *nullhomotopic* if $f \simeq *$. Thus $X \in \mathcal{T}$ is contractible if and only if id_X is nullhomotopic. For $f \in \mathcal{T}(X \to Y)$, f is nullhomotopic $\iff$ f factors through $CX := X \wedge I$—called the *cone* over X—via the inclusion $X \in CX \iff f$ factors up to homotopy through a contractible space.

1.1.8. **Relative homotopy.** For $f, g \in \mathcal{T}_*(X \to Y)$ and $A \subseteq X$, we say f is *homotopic to g rel A*, written $f \simeq g \operatorname{rel} A$, if $f|_A = g|_A$, $H\colon f \simeq g$, and $H(a, t) = f(a) = g(a)$ for all

$(a,t) \in A \times I$. We write $H\colon f \simeq g \operatorname{rel} A$ to indicate the relative homotopy. We say that A is *fixed* during the homotopy or that the homotopy is *stationary* on A.

Observe that there is no difference between $f \simeq g$ and $f \simeq g \operatorname{rel} \{*\}$ where $* \in X$ is the basepoint.

A special case of homotopy $\operatorname{rel} A$ occurs when $i\colon A \to X$ is an inclusion map, $r\colon X \to A$ is a retraction, and $ir \simeq \mathrm{id}_X \operatorname{rel} A$. Then r is called a *strong deformation retraction*, though some authors refer to a strong deformation retraction as a deformation retraction.) These notions also exist for unpointed spaces and unpointed maps.

1.1.9. **Homotopy of pairs and triples.** A *homotopy between maps of pairs* $f, g\colon (X,A) \to (Y,B)$ in $\mathcal{T}^{(2)}$ is a map of pairs $H\colon X \times I \to Y$ such that $H_0 = f$, $H_1 = g$, and $H(A \times I) \subseteq B$. We write $H\colon f \simeq g$ to denote H is a homotopy of maps of pairs from f to g, and we write $f \simeq g$ to indicate such a homotopy exists. We define

$$[(X,A),(Y,B)] \coloneqq \mathcal{T}^{(2)}\big((X,A) \to (Y,B)\big)/\simeq.$$

From here the generalization to homotopy of triples (and really for n-tuples for any $n \geq 3$) should be evident.

1.1.10. **Homotopy groups.** For $n \in \mathbb{Z}_{\geq 0}$ $X \in \mathcal{T}_*$ and $* \in X$, define $\mathbf{1} \coloneqq (1,0,\ldots,0) \in S^n$ and let

$$\pi_n(X) \coloneqq \pi_n(X,*) \coloneqq [(S^n,\mathbf{1}),(X,*)] = [(I^n,\partial I^n),(X,*)] \tag{1.1.10.1}$$

where $I^n \coloneqq I^{\times n}$ is the *n-dimensional box*, i.e., the n-fold product space of the unit interval I. This is a group if $n \geq 1$, and is abelian group if $n \geq 2$. Observe that $\pi_n(X) = \pi_{n-1}(\Omega X) = \cdots = \pi_0(\Omega^n X)$.

1.1.11. **Relative homotopy groups.** More generally, for $* \in A \subseteq X \in \mathcal{T}_*$, define the *relative homology groups* of the pair (X,A) by

$$\pi_n(X,A) \coloneqq \pi_n(X,A,*) \coloneqq [(D^{n+1},S^n,\mathbf{1}),(X,A,*)] = [(I^n,\partial I^n,J^n),(X,A,*)]$$

where $J^1 \coloneqq \{0\} \subseteq I$, and $J^n = (\partial I^{n-1} \times I) \cup (I^{n-1} \times \{0\}) \subseteq I^n$ for $n \geq 2$. Equivalently, $\pi_n(X,A) = \pi_{n-1}P(X;*,A)$ where $P(X;*,A)$ is the space of paths in X that begin at the basepoint and end in A. This is a group if $n \geq 2$ and an abelian group if $n \geq 3$. Again, $\pi_n(X,A) = \pi_0\left(\Omega^{n-1}P(X;*,A)\right)$.

1.2. CW COMPLEXES

1.2.1. **(Relative) CW complexes.** A *relative CW complex* is a pair $(X,A) \in \mathcal{T}^{(2)}$ together with a filtration $A = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq \cdots \subseteq X$ in $\mathcal{T}$ satisfying the following conditions.

(i) For all $n \geq 0$, X^n sits in the following pushout diagram.

$$\begin{tikzcd}
\coprod_{\alpha\in\Sigma_n} S^{n-1}_\alpha \arrow[r, hook] \arrow[d, "\coprod_{\alpha\in\Sigma_n}\varphi^n_\alpha"'] & \coprod_{\alpha\in\Sigma_n} D^n_\alpha \arrow[d, "\coprod_{\alpha\in\Sigma_n}\Phi^n_\alpha"] \\
X^{n-1} \arrow[r, hook] \arrow[ur, phantom, "\ulcorner", very near end] & X^n
\end{tikzcd}$$

(ii) $X = \operatorname{colim}_{n\to\infty}(A \hookrightarrow X_0 \hookrightarrow X_1 \hookrightarrow \cdots)$ in $\mathcal{T}$.

We call X^n the *n-skeleton* of (X, A). We call $X \in \mathcal{T}$ a *CW complex* if $(X, \varnothing)$ is a relative CW complex. We call the interiors $e^n_\alpha := (D^n_\alpha)^\circ$ *n-cells*, which are indexed by the sets Σ_n, which is also written $\Sigma_n(X, A)$ when the relative CW complex (X, A) is not clear. We call the maps φ^n_α and Φ^n_α the *attaching maps* and *characteristic maps* respectively for the corresponding n-cells, respectively. The *dimension* of (X, A) is n if $X = X^n$ and ∞ if there is no such n. A *subcomplex* of a CW complex X is a subspace $A \subseteq X$ that is a CW complex such that the characteristic maps $\Phi^n_\alpha\colon D^n_\alpha \to A$ compose with the inclusion $A \hookrightarrow X$ to characteristic maps in X, i.e., A is the union of some of the cells in X. (Thus the pair (X, A) can be viewed as a relative CW complex.)

1.2.2. **Unpacking the definition of CW complex.** Unpacking this definition, we see that an n-dimensional CW complex is a space $X \in \mathcal{T}$ consisting of subspaces $X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots \subseteq X^{n-1} \subsetneq X^n = X$, where the X^k, called the *k-skeleton* of X, satisfy the following properties.

(i) X^0 is a discrete set.

(ii) X^n is obtained from X^{n-1} by attaching n-disks D^n_α to X^{n-1} along continuous maps $\varphi_\alpha\colon \partial D^n_\alpha \cong S^{n-1} \to X^{n-1}$ in the sense that $X^n \cong (X^{n-1} \coprod_\alpha D^n_\alpha)/{\sim}$, where the equivalence relation $\sim$ is generated by $x \sim \varphi_\alpha(x)$ for all $x \in \partial D^n_\alpha$. Note that, as a set, $X^n = X^{n-1} \cup \coprod_{\alpha\in\Sigma_n} e^\alpha_n$.

(iii) If $n = \infty$, then X is the colimit of the filtration $A \hookrightarrow X^0 \hookrightarrow \cdots$, which gives the set $X = \bigcup_{n\geq 0} X^n$ equipped *weak topology*, which has a subset $A \subseteq X$ being open in X if and only if $A \cap X^n$ is open in X^n for all $n \geq 0$.

1.2.3. **Examples of CW complexes.**

(i) A simple graph $G = (V, E)$ is a CW complex of dimension 1.

(ii) A polygon with edge identifications is a CW complex of dimension 2.

(iii) There can be more than one CW complex structure on a topological space. For example, we can equip S^n with two different CW complex structures, namely

$$\mathrm{pt} = (S^n)^0 = (S^n)^1 = \cdots = (S^n)^{n-1} \subsetneq (S^n)^n = S^n \cong (\mathrm{pt} \amalg D^n)/(\partial D^n \sim \mathrm{pt})$$

or, for instance in the case of S^2, we can instead consider the CW structure

$$\mathrm{pt} = (S^2)^0 \subseteq (S^2)^1 \subseteq (S^2)^2 = S^2 \cong ((S^2)^1 \amalg D^2_+ \amalg D^2_-)/{\sim} = (D^2_+ \cup_{S^1 = \text{ equator}} D^2_-).$$

(iv) Real projective space $\mathbb{RP}^n$ of dimension n is an n-dimensional CW complex. Write $\mathbb{RP}^n \cong S^n/(x \sim -x)$. We first write down a $\mathbb{Z}/(2)$-equivariant CW structure on S^n, i.e., that as a set, we have $S^n = e^0_+ \cup e^0_- \cup e^1_+ \cup e^1_- \cup \cdots \cup e^n_+ \cup e^n_-$, so $\mathbb{RP}^n = e^0 \cup e^1 \cup e^2 \cup \cdots \cup e^n$. We claim $\mathbb{RP}^k$, identified as a subspace of $\mathbb{RP}^n$, is the k-skeleton of $\mathbb{RP}^n$, which is motivated by the following diagram.

$$\begin{tikzcd}
S^0 \arrow[r, phantom, "\subseteq"] \arrow[d, two heads] & S^1 \arrow[r, phantom, "\subseteq"] \arrow[d, two heads] & \cdots \arrow[r, phantom, "\subseteq"] & S^n \arrow[d, two heads] \\
\mathbb{RP}^0 \arrow[r, phantom, "\subseteq"] & \mathbb{RP}^1 \arrow[r, phantom, "\subseteq"] & \cdots \arrow[r, phantom, "\subseteq"] & \mathbb{RP}^n
\end{tikzcd}$$

It suffices to show the following diagram commues.

$$\begin{tikzcd}
\mathbb{RP}^n & & (\mathbb{RP}^{n-1} \amalg D^n)/{\sim} \arrow[ll, "h", "\cong"'] \\
& & \mathbb{RP}^{n-1} \amalg D^n \arrow[u, two heads] \arrow[ull, two heads, "i \amalg \Phi"]
\end{tikzcd}$$

Here $\Phi\colon D^n \to \mathbb{RP}^n$ is the characteristic map of the n-cell we are attaching, given by

$$\Phi(x_0, \dots, x_{n-1}) \coloneqq [x_0, \dots, x_{n-1}, \sqrt{1 - \|x\|^2}],$$

where $x = (x_0, \dots, x_{n-1})$. We take the attaching map φ to be the restriction $\varphi \coloneqq \Phi|_{\partial D^n}\colon \partial D^n = S^n \twoheadrightarrow \mathbb{RP}^n$, which is just the standard projection. This gives a well-defined bijection map h in the above diagram. Since $i \amalg \Phi$ is continuous, h is continuous. And h^{-1} is continuous since the domain is compact and the codomain is Hausdorff. Hence h is a homeomorphism, as desired.

1.2.4. **The HELP property.** The following generalizes the HEP and HLP. The *homotopy extension and lifting property* (HELP) is the property that if for all f, g, and p, the following square commutes up to a homotopy H, then there is a map $\widetilde{f}$ making the upper triangle strictly commute and making the bottom triangle commute up to a homotopy $\widetilde{H}$ extending H.

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "i"'] & E \arrow[d, "p"] \\
X \arrow[r, "f"'] \arrow[ur, dashed] & B
\end{tikzcd}$$

One recovers the HEP by considering $i\colon A \to X$, $E = B$, and $p = \mathrm{id}_B$ for all B, g, f. One recovers the HLP by considering $p\colon E \to B$ for $A = \varnothing$ for all X, f.

1.2.5. **The HELP lemma.**

Lemma 1.2.5.1 (HELP lemma)**.** *For a relative CW complex (X, A) of dimension $\leq n$, an*

n-equivalence $e\colon Y \to Z$, and a square

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & Y \arrow[d, "e"] \\
X \arrow[r] \arrow[ur, dotted] & Z
\end{tikzcd}$$

commuting up to a homotopy H, there exists a lift $X \to Y$ making the upper triangle strictly commute and making the lower triangle commute up to a homotopy $\widetilde{H}$ that extends H. That is, the HELP property is satisfied.

The HELP lemma thus states the following for (X, A) and $e\colon Y \to Z$ as above. Given maps $f : X \to Z, g : A \to Y$, and $H : A \times I \to Z$ such that $f|_A = h \circ i_0$ and $e \circ g = h \circ i_1$ in the following diagram, there are maps $\tilde{g}$ and $\tilde{H}$ that make the entire diagram commute.

$$\begin{tikzcd}
A \arrow[rr, hook, "\iota_0"] \arrow[dd] & & A \times I \arrow[dl, "H"'] \arrow[dd] & & A \arrow[ll, hook', "\iota_1"'] \arrow[dl, "g"'] \arrow[dd] \\
& Z & & Y \arrow[ll, "e"'] & \\
X \arrow[rr, hook, "\iota_0"'] \arrow[ur, "f"] & & X \times I \arrow[ul, dashed, "\widetilde{H}"] & & X \arrow[ll, hook', "\iota_1"] \arrow[ul, dashed, "\widetilde{g}"']
\end{tikzcd}$$

1.2.6. **Whitehead's theorem.** Informally, Whitehead's theorem says that the homotopy information of essentially all spaces X is encoded in the homotopy groups $\{\pi_n(X)\}_{n\geq 0}$. We write $\langle X, Y\rangle$ for homotopy classes of unpointed maps.

Here we follow [May11, §10.3], adapting the notation for our purposes. We have the following direct and important application of the HELP lemma.

Theorem 1.2.6.1 (Whitehead's theorem)**.**

(i) *If X is a CW complex and $e : Y \to Z$ is an n-equivalence, then $e_* : \langle X, Y\rangle \to \langle X, Z\rangle$ is a bijection if $\dim X < n$ and a surjection if $\dim X = n$.*

(ii) *If X and Y are weakly equivalent CW complexes, then X and Y are homotopy equivalent.*

Proof. (i): Suppose X is a CW complex and $e\colon Y \to Z$ is an n-equivalence. If $f \in \langle X, Z\rangle$, then by applying the HELP lemma to the pair $(X, \varnothing)$ we obtain a homotopy lift $\xi\colon X \to Y$ such that $e_*\langle\xi\rangle = \langle f\rangle$. Thus e_* is surjective. For injectivity, suppose e_* maps $\langle\xi\rangle$ and $\langle\xi'\rangle$ to $\langle f\rangle$. We want a homotopy $\xi \simeq \xi'$. Let $i\colon X \times \partial I \hookrightarrow X \times I$ be the inclusion. The square in the following diagram commutes up to the constant homotopy $H\colon f \circ i \simeq e \circ (\xi \amalg \xi')$.

$$\begin{tikzcd}
X \times \partial I \arrow[r, "\xi \amalg \xi'"] \arrow[d, hook, "i"'] & Y \arrow[d, "e"] \\
X \times I \arrow[r, "(x,t)\mapsto f(x)"] \arrow[ur, dashed, "\exists\,\eta"] & Z
\end{tikzcd}$$

Then, since $\dim X \le n$ implies $\dim(X \times I, X \times \partial I) \le n-1$, we can apply the HELP lemma on the pair $(X \times I, X \times \partial I)$ to obtain a map $\eta\colon X \times I \to Y$ such that $\eta \circ i = \xi \amalg \xi'$. From the facts $i = i_0 \amalg i_1$ for the inclusions $i_0\colon X \times \{0\} \hookrightarrow X$ and $i_1\colon X \times \{1\} \to X$, $(\xi \amalg \xi') \circ i_0 = \xi$, and $(\xi \amalg \xi') \circ i_1 = \xi'$, the following diagram commutes, so $\xi \simeq \xi'$ and thus e_* is injective.

$$\begin{tikzcd}
X \arrow[r, "\iota_0 = i \circ i_0"] \arrow[dr, "\xi"'] & X \times I \arrow[d, "\eta"] & X \arrow[l, hook', "\iota_1 = i \circ i_1"'] \arrow[dl, "\xi'"] \\
& Y &
\end{tikzcd}$$

(ii): We showed above that e_* is a bijection $\langle Y, Z\rangle \to \langle Z, Z\rangle$, so there is a unique $\langle f\rangle \in \langle Y, Z\rangle$ such that $\langle e \circ f\rangle = \langle \mathrm{id}_Z\rangle$. Thus $e \circ f \simeq \mathrm{id}_Z$. Similarly, we have a bijection $e_*\colon \langle Y, Y\rangle \to \langle Z, Y\rangle$, so we can apply e_*^{-1} to both sides of the equation $e_*\langle f \circ e\rangle = \langle e \circ f \circ e\rangle = \langle \mathrm{id}_Z \circ e\rangle = e_*\langle \mathrm{id}_Y\rangle$ to obtain $\langle f \circ e\rangle = \langle \mathrm{id}_Y\rangle$, which gives $f \circ e \simeq \mathrm{id}_Y$. Thus $Y \simeq Z$. □

Corollary 1.2.6.2. *A connected CW complex X is contractible if and only if $\pi_q X = 0$ for each $k \ge 1$.*

If X is a *finite* CW complex in the sense that it has finitely many cells, and if $\dim X > 1$ and X is not contractible, then it is known that X has infinitely many nonzero homotopy groups. Whitehead's theorem is thus surprisingly strong: by its first statement, if low-dimensional homotopy groups are mapped isomorphically, then so are all higher homotopy groups.

1.2.7. **Cellular approximation of maps.** For CW complexes X and Y, a map $f\colon X \to Y$ is *cellular* if $f(X^n) \subseteq Y^n$ for all $n \ge 0$.

Theorem 1.2.7.1 (Cellular approximation theorem)**.** *For CW complexes X and Y, every map $f\colon X \to Y$ is homotopic to a cellular map. If f is already cellular on a subcomplex A of X, then we can choose the homotopy to restrict to the constant homotopy on A.*

Proof. For the base case $n = 0$, note that for every 0-cell $x_\alpha \in X^0$, $f(x_\alpha)$ lies in the same path component as some 0-cell Y^0. For each $\alpha \in \Sigma_0(X)$, choose such a 0-cell y_α and a path $\gamma_\alpha\colon I \to Y$ from $f(x_\alpha)$ to y_α and define $H\colon X \times I \to Y$ by

$$H_t(x) \coloneqq \begin{cases} \gamma_\alpha(t) & \text{if } x = x_\alpha \text{ for some } \alpha \in \Sigma_0(X), \\ f(x) & \text{otherwise.} \end{cases}$$

We claim H is continuous. Since the components of $X \times I = ((X \smallsetminus X_0) \times I) \cup (X^0 \times I) = X \smallsetminus \bigcup_{\alpha \in \Sigma_n}(\{x_\alpha\} \times I)$ are disjoint closed sets, it suffices to check the restrictions $H|_{X \smallsetminus X^0}$ and $H|_{\{x_\alpha\} \times I}$ for $\alpha \in \Sigma_0(X)$ are continuous. The former restriction is just $(x, t) \mapsto f(x)$, which is continuous. The latter restrictions are $(x_\alpha, t) \mapsto \gamma_{x_0}(t)$, which is also continuous, so H is continuous. Write $f_0 \coloneqq H^1$. Then $f \simeq f_0$ and $f(X^0) \subseteq Y^0$, and the homotopy H is stationary on A because A is disjoint from X^0, which proves the base case.

For the induction step, suppose we have a homotopy $H\colon f \simeq f_n \operatorname{rel} A$ for some map $f_n\colon X \to Y$ such that $f_n|_{X_n}$ is cellular, i.e., $f_n(X^k) \subseteq Y^k$ for all $k \leq n$. We seek a map $f_{n+1}\colon X \to Y$ such that $f_{n+1}|_{X^{n+1}}$ is cellular and $f \simeq f_{n+1} \operatorname{rel} A$.

For $\alpha \in \Sigma_{n+1}(X)$, consider attaching map $\varphi_\alpha^{n+1}\colon S_\alpha^n \to X^n$. This gives a homotopy $H \circ \varphi_\alpha^{n+1}\colon f \circ \Phi_\alpha^{n+1} \circ i \simeq j \circ f_n \circ \varphi_\alpha^{n+1} \operatorname{rel} A$, where $i\colon S_\alpha^n \hookrightarrow D_\alpha^{n+1}$ and $j\colon Y^n \hookrightarrow Y$ are inclusions, i.e., the outer square in the following diagram commutes up to the homotopy $H \circ \varphi_\alpha^{n+1}$.

$$\begin{tikzcd}
S_\alpha^n \arrow[r, "f_n\circ\varphi_\alpha^{n+1}"] \arrow[d, hook, "i"'] & Y^n \arrow[d, hook, "j"] \\
D_\alpha^{n+1} \arrow[r, "f_n\circ\Phi_\alpha^{n+1}"] \arrow[ur, dashed, "\exists g"] & Y^{n+1}
\end{tikzcd}$$

We now apply the HELP lemma to the pair (Y^n, Y^{n+1}) to obtain a lift $g\colon D_\alpha^{n+1} \to Y^n$ making the top triangle strictly commute, i.e., $g|_{S_\alpha^{n+1}} = f_n \circ \varphi_\alpha^{n+1}$, and an extension $\widetilde{H}$ to the homotopy H such that $\widetilde{H}\colon j \circ g \simeq f_n \circ \Phi_\alpha^{n+1} \operatorname{rel} A$, proving the result. Since α was an arbitrary element of $\Sigma_n(X)$, we are done. □

The same proof extends (without any new ideas) to show that if $f\colon (X, A) \to (Y, B)$ is a map of relative CW complexes, then f is homotopic $\operatorname{rel} A$ to a cellular map.

1.2.8. **Approximation of spaces and pairs by CW complexes.** Here we will see that for every space X, there is a CW complex Z and a weak homotopy equivalence $Z \to X$, called a *CW approximation* of X. Since homotopy groups are invariant (by definition) under weak homotopy equivalence, by Whitehead's theorem this will show that every space is weakly equivalent to some CW complex. See [May11, §10.5] for approximation of spaces and see [May11, §10.5] for approximation of pairs.

The following result says that there is a functor $\Gamma : \mathsf{Ho}\,\mathcal{T} \to \mathsf{Ho}\,\mathcal{T}$ and a natural transformation $\gamma : \Gamma \to \mathrm{id}_{\mathsf{Ho}\,\mathcal{T}}$ that assign a CW complex ΓX and a weak homotopy equivalence $\gamma : \Gamma X \to X$ to a space X.

Theorem 1.2.8.1 ([May11, Theorem 10.5, adapted]). *There is a functor* $\Gamma\colon \mathsf{Ho}\,\mathcal{T} \to \mathsf{Ho}(\mathcal{T}^{\mathrm{CW}})$ *and a natural transformation* $\gamma\colon \Gamma \Rightarrow \mathrm{id}_{\mathsf{Ho}\,\mathcal{T}}$ *whose component morphisms are weak equivalences compatible with* Γ *in the sense that for all* f, Γf *is the unique morphism in* $\mathsf{Ho}\,\mathcal{T}$ *making the following diagram commute up to homotopy.*

$$\begin{tikzcd}
\Gamma X \arrow[r, "\Gamma f"] \arrow[d, "\gamma_X"'] & \Gamma Y \arrow[d, "\gamma_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

If X is n-connected for some $n \geq 1$, then ΓX can be chosen to have a unique vertex (0-cell) and no q-cells for $1 \leq q \leq n$. There is a generalization for pairs; see [May11, Theorem 10.6].

We will use the following lemma, which will be useful in its own right.

Lemma 1.2.8.2 ([Str11, Problem 4.66, adapted])**.** *For any space $X \in \mathcal{T}_*$, there is a wedge sum of spheres $W = \bigvee_\alpha S^{n_\alpha}$ and a map $w \colon W \to X$ such that the induced map $w_* \colon \pi_n(W) \to \pi_n(X)$ is surjective for every n.*

Proof. Fix $X \in \mathcal{T}_*$, and for each $n \geq 0$, let $\{[w_\alpha \colon S^n \to X]\}_{\alpha \in I}$ be a set of generators for $\pi_n(X)$, and define $W \coloneqq \bigvee_{\alpha \in I, n \geq 0} S^n_\alpha$, where S^n_α denotes a copy of the sphere S^n corresponding to the index α. Now let $w \colon W \to X$ be the map determined on its summands S^n_α by w_α. Then w is surjective, since a generator $[w_\alpha \colon S^n \to X]$ is simply $w \circ \iota_\alpha$, where ι_α is the inclusion $[\iota_\alpha \colon S^n_\alpha \to X]$ of X, which is a generator of $\pi_n(X)$. □

Proof of Theorem 1.2.8.1. Let $X \in \mathcal{T}$. By working with each path-component at a time, we may assume X is path-connected and with with pointed spaces and pointed maps. Our plan is to define ΓX to be the colimit of a diagram of the following form such that i_k is cellular for each $k \geq 0$.

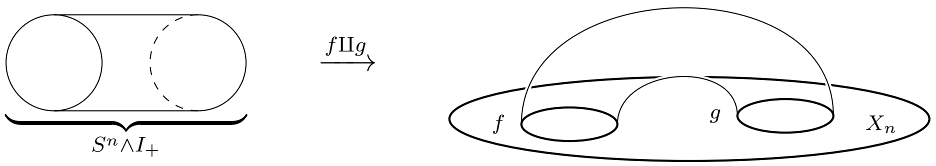
$$\begin{tikzcd}
X_1 \arrow[r, hook, "i_1"] \arrow[drr, "\gamma_1"'] & X_2 \arrow[r, hook, "i_2"] \arrow[dr, "\gamma_2"] & \cdots \arrow[r, hook] & X_n \arrow[r, hook, "i_n"] \arrow[dl, "\gamma_n"'] & X_{n+1} \arrow[r, hook] \arrow[dll, "\gamma_{n+1}"] & \cdots \\
 & & X & & &
\end{tikzcd} \tag{1.2.8.3}$$

By Lemma 1.2.8.2, there is a wedge sum of spheres X_1 and a map $\gamma_1 \colon X_1 \to X$ such that $\gamma_* \colon \pi_q(X_1) \to \pi_q(X)$ is surjective for all $q \geq 0$. We equip X_1 with the obvious CW structure.

Now, for the induction step, suppose for $1 \leq m \leq n$ we have CW complexes X_m, cellular inclusion maps $i_m \colon X_m \hookrightarrow X$, and maps $\gamma_m \colon X_m \to X$ such that $\gamma_m \circ i_{m-1} = \gamma_{m-1}$ and $(\gamma_m)_* \colon \pi_q(X_m) \to \pi_q(X)$ is a surjective for all q and a bijection for all $q \leq n-1$. Now define

$$X_{n+1} \coloneqq X_n \amalg_\sim \bigvee (S_n \wedge I_+),$$

where the wedge sum is indexed by the set of pairs $([f],[g])$ of cellular representatives $[f],[g] \in \pi_n(X)_n)$, $[f] \neq [g]$, and $(\gamma_n)_*[f] = (\gamma_n)_*[g]$, and $\sim$ is the equivalence relation generated by $(s,0) \sim f(s)$ and $(s,1) \sim g(s)$ for all $s \in S^n$. In short, this means that each indexed pair (f,g) is attached to X_n in the construction of X_{n+1} as follows.



In short, constructing X_{n+1} "patches the holes" in X_n, allowing homotopies to pass through the newly attached handles. Let $i_{n+1}\colon X_n \hookrightarrow X_{n+1}$ be the inclusion, which is a cellular map by construction and satisfies $(i_n)_*[f] = (i_n)_*[g]$ since we can homotope f to g in X_{n+1} by passing across the newly glued handle. Now define $\gamma_{n+1}\colon X_{n+1} \to X$ by choosing homotopies h on the cylinder and homotoping $\gamma_n \circ f$ to $\gamma_n \circ g$, so that $(\gamma_{n+1})_*\colon \pi_q(X_{n+1}) \to \pi_q(X)$ is a surjection for all q because $(\gamma_n)_*$ is. This is a bijection for all $q \leq n$ by construction. Note that we have not changed the homotopy groups in dimension $\leq n$.

Note that the cylinder $S^n \wedge I_+$ is a CW complex with subcomplex $S^n \wedge (\partial I)_+$ and f, g are cellular, so the pushout

$$\begin{tikzcd}
S^n_\alpha \arrow[r] \arrow[d, "{f_\alpha, g_\alpha}"'] \arrow[dr, phantom, "\ulcorner", very near end] & \coprod_\alpha D^{n+1}_\alpha \arrow[d, "{F_\alpha, G_\alpha}"] \\
X_n \arrow[r, hook] & X_{n+1}
\end{tikzcd}$$

implies X_{n+1} is a CW complex with X_n as a subcomplex.

Next we have the diagram (1.2.8.3), so we define

$$\Gamma X \coloneqq \operatorname{colim}\left(X_0 \overset{i_0}{\hookrightarrow} X_1 \overset{i_2}{\hookrightarrow} X_2 \overset{i_3}{\hookrightarrow} \cdots\right).$$

Since the maps $\{\gamma_n\}_{n\geq 0}$ form a cone under the sequence of which ΓX is a colimit, by the universal property of the colimit there is a unique comparison map $\gamma_X\colon \Gamma X \to X$ preserving the commutativity of (1.2.8.3). And ΓX is a CW complex by definition, as a colimit of a filtration of CW complexes of increasing finite dimension. Moreover, one finds $(\gamma_X)_*\colon \pi_q(\Gamma X) \to \pi_q(X)$ is an isomorphism for all $q \geq 0$.

Finally, if X is n-connected for $n \geq 1$, then ΓX has no q-cells for $q \leq n$ in the construction, since the index set of (f, g) is empty, forcing $X_q = X_{q-1} = \cdots = X_0$, and we defined X_0 to be a wedge sum of spheres with a single 0-cell, namely the basepoint. ☐

1.3. Covering space theory

TODO: Under construction See here.

1.3.1. **Covering spaces.**

The following result is useful when doing computations.

Proposition 1.3.1.1. *For a covering map* $p\colon \widetilde{X} \to X$, $p_*\colon \pi_n(\widetilde{X}, \widetilde{*}) \to \pi_n(X, p(\widetilde{*}))$ *is an isomorphism for all* $n \geq 2$.

1.3.2. **Path lifting property.**

1.3.3. **Deck transformations.**

1.3.4. Classification of covering spaces.

1.4. FIBER BUNDLES

1.3.5. Construction of the universal cover. A great resource for fiber bundles is here.

1.4.1. Trivial fiber bundles. In $\mathcal{T}$, a *trivial fiber bundle* over B with fiber F, or *trivial F-bundle*, is any map $p\colon E \to B$ such that a diagram of the form

$$\begin{tikzcd} B \times F \arrow[rr, "\cong"] \arrow[dr, "\mathrm{pr}_1"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

commutes, where pr_1 is projection onto the first factor.

1.4.2. Fiber bundles. A *fiber bundle* with fiber F, or an *F-bundle*, is a map $p\colon E \to B$ equipped with an open cover $\{U_\alpha\}_{\alpha\in I}$ of B, called a *local trivialization*, such that the restrictions $p_\alpha := p|_{E_\alpha}\colon E_\alpha := p^{-1}(U_\alpha) \to U_\alpha$ is a trivial F-bundle.

The shorthand to introduce a fiber bundle over B with fiber F is along the lines of "consider a fiber bundle $F \to E \xrightarrow{p} B$", and p is called the *projection map*. We call an element U_α of the local trivialization $\{U_\alpha\}_{\alpha\in I}$ a *trivializing neighborhood*. For a fiber bundle $F \to E \to B$, the *total space* is E, the *base space* is B, and the *fiber* is F. We often identify fiber bundles with their total space E.

Be warned that some authors (such as [Str11]) reserve the terminology "fiber bundle" for what we will call *G-structured fiber bundle* for some group G, opting to use and some (again [Str11]) say *G-bundle* to mean a fiber bundle with structure group G instead of a fiber bundle with fiber G. However, for principal G-bundles, these notions coincide and there is no confusion.

Lemma 1.4.2.1. *A fiber bundle over a paracompact base space is a fibration. Similarly, a countable fiber bundle is a fibration.*

For fiber bundles $p_1\colon E_1 \to B_1$ and $p_2\colon E_2 \to B_2$, a *bundle map* is a commutative square

$$\begin{tikzcd} E_1 \arrow[r, "g"] \arrow[d, "p_1"] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "f"] & B_2 \end{tikzcd}$$

Thus there is a category of fiber bundles. A *bundle isomorphism* is an isomorphism in this category. A *covering space* is a fiber bundle $F \to E \xrightarrow{p} B$ for which p is a local homeomorphism.

1.4.3. **Structure group of a fiber bundle.** Consider a fiber bundle $F \to E \xrightarrow{p} B$ and overlapping trivializing neighborhoods U_α and U_β in B. The restriction of a homeomorphism is a homeomorphism, so if we write $U_{\alpha\beta}$ to mean $U_\alpha \cap U_\beta$ and $E_{\alpha\beta}$ to mean $p^{-1}(U_{\alpha\beta})$, then we have the following commutative diagram.

$$\begin{tikzcd}
U_{\alpha\beta} \times F \arrow[r, "(\varphi_\alpha|_{E_{\alpha\beta}})^{-1}", "\cong"'] \arrow[dr, "\mathrm{pr}_1"'] & E_{\alpha\beta} \arrow[r, "\varphi_\beta|_{E_{\alpha\beta}}", "\cong"'] \arrow[d, "p"] & U_{\alpha\beta} \times F \arrow[dl, "\mathrm{pr}_1"] \\
 & U_{\alpha\beta} &
\end{tikzcd}$$

Thus $\Phi_{\alpha\beta} \coloneqq \varphi_\beta|_{E_{\alpha\beta}} \circ (\varphi_\alpha|_{E_{\alpha\beta}})^{-1} \colon U_{\alpha\beta} \times F \to U_{\alpha\beta} \times F$ is a homeomorphism such that $\mathrm{pr}_1 \circ \Phi_{\alpha\beta} = \mathrm{pr}_1$, so $\Phi_{\alpha\beta}(x, f) = (x, \mathrm{pr}_2 \circ \Phi^2_{\alpha\beta}(x, f))$. In particular $\Phi_{\alpha\beta}$ fixes the first component $U_{\alpha\beta}$ in the product $U_{\alpha\beta} \times F$, i.e., $\Phi_{\alpha\beta}(x, f) = (x, g_{\alpha\beta}(x)(f))$ for some $g_{\alpha\beta}(x)(f) \in F$. Since $\Phi_{\alpha\beta}$ is a homeomorphism, this implies the assignment $g_{\alpha\beta}(x) \colon F \to F$ given by $f \mapsto g_{\alpha\beta}(x)(f)$ is a homeomorphism. We call the assignments of this homeomorphism depending on the points $x \in U_{\alpha\beta}$ the *transition function* for the overlap. We call a group G a *structure group* for the fiber bundle if F is a G-space such that for any overlapping trivializing neighborhoods U_α and U_β and any $x \in U_{\alpha\beta}$, $g_{\alpha\beta}(x)(f) = g \cdot f$ for some $g \in G$. In this case we call the fiber bundle *G-structured*. Every fiber bundle $F \to E \to B$ has such a structure, since for example the group $\mathrm{Homeo}(F)$ of homeomorphisms of F acts on F in the obvious way and satisfies the required axiom.

1.4.4. **Fiber bundle construction lemma.** The transition functions $\{g_{\alpha\beta} \colon U_{\alpha\beta} \to G\}_{\alpha,\beta\in I}$ of a G-structured fiber bundle $F \to E \xrightarrow{p} B$ with local trivialization $\{U_\alpha\}_{\alpha\in I}$ satisfy the following conditions for all points x in the relevant intersections among trivializing neighborhoods U_α, U_β, and U_γ.

(i) $g_{\alpha\alpha}(x) = 1_G$.
(ii) $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)^{-1}$.
(iii) (*Cocycle condition*) $g_{\alpha\gamma}(x) = g_{\alpha\beta}(x) g_{\beta\gamma}(x)$.

We can use transition functions to construct a fiber bundle when we are only given the base space B and fiber F. It turns out these data characterize the fiber bundle. The proof of the following theorem constructs the desired fiber bundle. See here for the details.

Theorem 1.4.4.1 (Fiber bundle construction theorem). *For any topological spaces B and F such that*

(i) *F is a left G-space,*
(ii) *$\{U_\alpha\}_{\alpha\in I}$ is an open cover of B,*
(iii) *$\{g_{\alpha\beta} \colon U_{\alpha\beta} \to G\}_{\alpha,\beta\in I}$ are continuous maps satisfying the cocycle condition, i.e., $g_{\alpha\gamma}(x) = g_{\alpha\beta}(x) g_{\beta\gamma}(x)$ for all $x \in U_{\alpha\beta\gamma}$ and all $\alpha, \beta, \gamma \in I$,*

then there exists a G-structured fiber bundle $F \to E \xrightarrow{p} B$ with local trivialization $\{U_\alpha\}_{\alpha\in I}$ and transition functions $\{g_{\alpha\beta}\}_{\alpha,\beta\in I}$.

1.4.5. **Principal bundles.** The data of an fiber bundle with fiber F in terms of transition functions lets us define an F' bundle for any other G-space F. In particular, we can choose $F' = G$, which is a G-space with action just left (or right) multiplication in G. We call any fiber bundle that can be constructed in this way a *principal G-bundle.*

1.4.6. **Obtaining principal bundles from free G-spaces.**

Proposition 1.4.6.1. *For a free G-space E, if the quotient map $p\colon E \to E/G$ admits local sections, then p is a principal G-bundle.*

1.4.7. **Construction of principal G-bundles.** **TODO: See p. 10 here. Alternatively, read [Str11, §15.3] and do all the exercises therein.**

1.4.8. **Change of fibers.** See here, §1.16. **TODO: See [Str11, p. 335]**

1.4.9. **Associated bundles for principal G-bundles.** KEY EXAMPLE: GL_n acts on any local frame of the frame bundle on a smooth manifold, i.e., on the bundle of local frames on the manifold. The associated bundle of this action gives a principal GL_n-bundle, etc. **TODO: add, fix, clean, etc.**

TODO: Notebook. TODO: See [Str11, p. 335]

1.4.10. **Vector bundles.** An *n-dimensional vector bundle* is a fiber bundle with fiber $F := \mathbb{R}^n$ in the usual Euclidean topology such that the local trivializations $p_\alpha\colon E_\alpha \to U_\alpha$ restrict to linear isomorphisms on fibers. A *line bundle* is a 1-dimensional vector bundle. A *morphism* of vector bundles over a fixed base space is a morphism as in the category defined for fiber bundles with bottom map an equality, that is, a continuous map $g\colon E_1 \to E_2$ that makes the following diagram commute, and g is a *bundle isomorphism* if in addition g is a homeomorphism.

$$\begin{tikzcd}
E_1 \arrow[rr, "g"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\
& B &
\end{tikzcd}$$

By replacing $\mathbb{R}^n$ with $\mathbb{C}^n$, we obtain the definition for an *n-dimensional complex vector bundle*, and all subsequent notions have complex analogs. We assume vector bundles are real unless otherwise stated.

Examples of vector bundles include tangent bundles, normal bundles, the Möbius bundle, canonical (also called tautological) line bundles for real projective spaces and more generally for Grassman manifolds, and more.

A *section* of a vector bundle $p\colon E \to B$ is a map $s\colon B \to E$ such that $ps = 1$, i.e., such that $s(b) \in p^{-1}(b)$. Every vector bundle has a canonical section with value 0, namely the *zero section*, which we identify with its image, a subspace of E which p projects homeomorphically onto B.

Lemma 1.4.10.1. *An n-dimensional vector bundle $p\colon E \to B$ is isomorphic to the trivial bundle if and only if there are n sections $s_1, \dots, s_n\colon B \to E$ such that the vectors $s_1(b), \dots, s_n(b)$ are linearly independent in each fiber $p^{-1}(b)$.*

Proof. ($\Rightarrow$) The trivial bundle $B \times \mathbb{R}^n$ has linearly independent sections $s_i(b) = e_i \in p^{-1}(b)$ where e_i is the ith standard ordered basis of $\mathbb{R}^n$, and linear isomorphisms preserve linearly independent sets.

($\Leftarrow$) If such sections exist, then the map $h\colon B \times \mathbb{R}^n \to E$ given by $h(b, t_1, \dots, t_n) = \sum_i t_i s_i(b)$ is a linear isomorphism in each fiber, is is continuous because its composition with a local trivialization $p^{-1}(U) \to U \times \mathbb{R}^n$ is continuous. Thus h is an isomorphism by the following lemma. ☐

Lemma 1.4.10.2 ([Hat17, Lemma 1.1])**.** *A continuous map $h\colon E_1 \to E_2$ between vector bundles over the same base space B is an isomorphism if it takes each fiber $p_1^{-1}(b)$ to the corresponding fiber $p_2^{-1}(b)$ by a linear isomorphism.*

1.4.11. **Direct sums of vector bundles.** Define the *direct sum* (or *Whitney sum*) of fiber bundles $p_1\colon E_1 \to B$ and $p_2\colon E_2 \to B$ by the $p_1 \oplus p_2\colon E_1 \oplus E_2 \to B$ constructed by the following pullback diagram.

$$\begin{tikzcd}
E_1 \oplus E_2 \arrow[r] \arrow[d] & E_2 \arrow[d, "p_2"] \\
E_1 \arrow[r, "p_1"] & B
\end{tikzcd}$$

Thus $E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 \mid p_1(v_1) = p_2(v_2)\}$. One can check the fibers of $E_1 \oplus E_2$ are the direct sums of fibers of E_1 and E_2.

The following paragraph is from [Hat17, p. 10]. The direct sum of trivial bundles is trivial, but the direct sum of nontrivial bundles may be trivial. For instance, the direct sum of the tangent and normal bundles to S^n in $\mathbb{R}^{n+1}$ is the trivial bundle $S^n \times \mathbb{R}^{n+1}$ since elements of the direct sum are triples $(x, v, tx) \in S^n \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ with $x \perp v$ and the map $(x, v, tx) \mapsto (x, v + tx)$ gives an isomorphism of the direct sum bundle with $S^n \times \mathbb{R}^{n+1}$. So the tangent bundle to S^n is *stably trivial*: it becomes trivial after taking the direct sum with a trivial bundle.

1.4.12. **Inner products of vector bundles.** An *inner product* on a vector bundle $p\colon E \to B$ is a map $\langle -|-\rangle : E \oplus E \to \mathbb{R}$ that restricts to an inner product in each fiber, i.e., to a positive-definite symmetric bilinear form.

Proposition 1.4.12.1 ([Hat17, Proposition 1.2]). *A (Hermitian) positive-definite inner product exists for a (complex) vector bundle $p\colon E \to B$ if B is paracompact.*

Proposition 1.4.12.2 ([Hat17, Proposition 1.3]). *If $E \to B$ is a vector bundle over a paracompact space B and $E_0 \subseteq E$ is a vector subbundle, then there i a vector subbundle $E_0^{\perp} \subseteq E$ over B such that $E_0 \oplus E_0^{\perp} \cong E$ as bundles.*

Proposition 1.4.12.3 ([Hat17, Proposition 1.4]). *For each vector bundle $E \to B$ over a compact Hausdorff space B, there exists a vector bundle $E' \to B$ such that $E \oplus E'$ is a trivial bundle.*

1.4.13. **Tensor products of vector bundles.** For vector bundles $p_1\colon E_1 \to B$ and $p_2\colon E_2 \to B$, there is a *tensor product* bundle $p_1 \otimes p_2\colon E_1 \otimes E_2 \to B$ constructed in [Hat17, p. 14].

1.4.14. **Exterior and symmetric powers of bundles.** The tensor product of bundles allows us to consider the exterior powers and symmetric powers of fibers in bundles. See [Hat17, p. 15] for the details.

1.4.15. **Quotient bundles.** **TODO: Under construction.**

1.4.16. **Constructing fiber bundles from vector bundles.** Given a vector bundle $E \to B$, one may construct several "associated fiber bundles", whose details can be found on [Hat17, "Associated Fiber Bundles", p. 15]. In short, there is the *sphere bundle* $S(E) \to E$ obtained by factoring out scalar multiplication in each fiber, the *projective bundle* of one-dimensional subspaces of the fibers (topologized as the quotient of $E \to B$ with $S(E) \to E$), the *flag bundle* $F_k(E) \to E$ of k-tuples of orthogonal lines of fibers of E through the origin, the *Stiefel bundle* $V_k(E) \to B$ of k-tuples of orthogonal *unit* vectors in the fibers of E through the origin, and the *Grassman bundle* $G_k(E) \to B$ of k-dimensional linear subspaces of fibers. In the complex case, there is also the *conjugate bundle* (in the complex case) of the complex conjugates of each fiber.

Chapter 2

(Co)homology

2.1. (SINGULAR) HOMOLOGY

2.1.1. **Simplices in Euclidean space.** An *n-simplex* is the convex hull S of some $n+1$ points $v_0, \dots, v_n$ in $\mathbb{R}^m$; we write $S = [v_0, \dots, v_n]$. Points $s \in [v_0, \dots, v_n]$ are thus of the form $s = \sum_{j=0}^n t_j v_j$ for some $t_0, \dots, t_n \in [0,1]$ (called *barycentric coordinates*) such that $\sum_{j=0}^n t_j = 1$. The *standard n-simplex* Δ^n is the convex hull of $v_0, \dots, v_n$ in $\mathbb{R}^{n+1}$, where the v_j now and henceforth denote the standard basis vectors of $\mathbb{R}^{n+1}$. We write $[v_0, \dots, \widehat{v_i}, \dots, v_n]$ for the continuous map $\Delta^n \to \Delta^{n-1}$ given by $\sum_{j=0}^n t_j v_j \mapsto \sum_{i=0, j \neq i}^n t_j v_j$, i.e., the ith face (an $(n-1)$-simplex) of the n-simplex Δ^n. This is called the ith *collapse* map (or *degeneracy* or *degenerate* map), as it collapses the n-simplex onto its ith face.

Also note that there is the ith *inclusion* map (or *face* or *face inclusion* map) $\Delta^{n-1} \to \Delta^n$ given by including the $(n-1)$-simplex into Δ^n as its ith face.

2.1.2. **(Singular) homology.** Fix $X \in \mathsf{Top}$. A *singular n-simplex* in X is a continuous map $\sigma \colon \Delta^n \to X$. Define the *singular chain complexes* $C_\bullet(X)$ at degree n by the free abelian group generated by the singular n-simplices, i.e., $C_n(X) \coloneqq \mathbb{Z}[\mathrm{Map}(\Delta^n, X)]$. Following through this reasoning in general, in the end, we get for a singular simplex in Δ^n that $\partial[v_0, \dots, v_n] \coloneqq \sum_{i=0}^n (-1)^i [v_0, \dots, \widehat{v_i}, \dots, v_n]$, which is a linear combination of $(n-1)$-simplices in Δ^{n-1}. Now $\partial(\sigma \colon \Delta^n \to X) \coloneqq \sum_{i=0}^n (-1)^i \sigma \circ [v_0, \dots, \widehat{v_i}, \dots, v_n]$ is the composition $X \xleftarrow{\sigma} \Delta^n \xleftarrow{\partial} \Delta^{n-1}$, and extending to linear combinations this gives us a map $\partial_n \colon C_n(X) \to C_{n-1}(X)$. The data of free abelian groups generated by singular simplices in X, together with their boundary maps, is written $(C_\bullet(X), \partial_\bullet)$, or sometimes just $C_\bullet(X)$ or $C_\bullet$ when clear from context. One proves the following by several index shifts and cancellations of sums.

Theorem 2.1.2.1. *For $X \in \mathsf{Top}$, the composition $\partial_n \circ \partial_{n+1} \colon C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(x)$ is zero. Thus $(C_\bullet(X), \partial_\bullet))$ is a chain complex, called the* singular chain complex of

X.

Define the *singular homology* of X by $H_k(X) := H_k(C_\bullet(X))$.

Lemma 2.1.2.2 (Homology of a point)**.** *For the single point space* pt, $H_k(\mathrm{pt}) = \mathbb{Z}\delta_{k=0}$.

Proof. The only 0-simplex is the constant map $\Delta^0 \to \mathrm{pt}$, so $C_0 = \mathbb{Z}[\{\sigma_0\}] = \mathbb{Z}$ and thus $H_0(\mathrm{pt}) = Z_0/B_0 = C_0/0 = C_0 = \mathbb{Z}$. For $k \geq 1$ and a map $\sigma_{k-1}\colon \Delta^k \to X$, $\partial\sigma_k = \sum_{i=0}^{k}(-1)^i\sigma_k \circ [v_0, \ldots, \widehat{v_i}, \ldots, v_k] = \sigma_{k-1}$ if k is even and 0 if k is odd, so $H_k(\mathrm{pt}) = Z_k/B_k = 0/B_k = 0$. □

Lemma 2.1.2.3. *If X is path-connected, then $H_0(X) \cong \mathbb{Z}$.*

Proof. The desired isomorphism $H_0(X) = C_0(X)/B_0(X) \to \mathbb{Z}$ is induced by the *augmentation homomorphism* $\varepsilon\colon C_0(X) \to \mathbb{Z}$, $\sum n_i x_i \mapsto \sum n_i$. Note $\varepsilon \circ \partial\colon C_1(X) \to C_0(X) \to \mathbb{Z}$ is zero, since $\varepsilon\partial(\sigma_1) = \varepsilon(\partial_0\sigma_1 - \partial_1\sigma_1) = \varepsilon(\sigma(v_1) - \sigma(v_0)) = +1 - 1 = 0$. Thus ε descends to a well-defined homomorphism $\overline{\varepsilon}\colon H_0(X) = C_0(X)/B_0(X) \to \mathbb{Z}$ making the following diagram commute.

$$\begin{tikzcd}
C_0(X) \arrow[r, two heads, "\varepsilon"] \arrow[d, two heads, "q"'] & \mathbb{Z} \\
C_0(X)/B_0(X) \arrow[ur, two heads, "\overline{\varepsilon}"'] &
\end{tikzcd}$$

For a point $\mathrm{pt} \in X$, $\overline{\varepsilon}(\mathrm{pt}) = 1$ is a generator for $\mathbb{Z}$, so $\overline{\varepsilon}$ is surjective. It is also injective since $\overline{\varepsilon}([\sum n_i x_i]) = \sum n_i$ is the trivial homology class: to see this, we need a 2-chain c with $\partial c = \sum n_i x_i$. Pick a base point x_0 and paths $\lambda_i\colon I \to X$ from x_0 to x_i. Considering λ_i as a 1-simplex, we have $\partial\lambda_i = \partial_0\lambda_i - \partial_1\lambda_i = x_i - x_0$, so for $c = \sum n_i x_i$, $\partial c = \partial(\sum n_i\lambda_i) = \sum n_i x_i - (\sum n_i)x_0 = \sum n_i x_i$ as desired. □

2.1.3. **Relating the first homology and the fundamental group.** Fix a space $X \in \mathsf{Top}_*$. Here we show $H_1(X) \cong \pi_1(X)^{\mathrm{ab}}$, the abelianization. We show this by defining a map $\pi_1(X) \to H_1(X)$ and showing it descends to an isomorphism $\pi_1(X)/[\pi_1(X), \pi_1(X)] = \pi_1(X)^{\mathrm{ab}} \to H_1(X)$. The obvious choice is to send equivalence classes $[\gamma]$ of based loops to themselves in $Z_1(X)$, which is a well-defined assignment since based loops in X singular 1-simplices that are cycles as they start and end at the same point. One then passes to the quotient $[[\gamma]] \in Z_1(X)/B_1(X) = H_1(X)$. This map is called the *Hurewicz map*, and one can show it is indeed a group isomorphism.

2.1.4. **Reduced homology.** It is natural to want all homologies of the single point space to vanish, so to do this we can slightly alter the property of dimensionality by considering a new but similar functor $\widetilde{H}_k$. Define the *reduced chain complex* of X by

$$\widetilde{C}_\bullet\colon \qquad \mathbb{Z} \xleftarrow{\varepsilon} C_0(X) \longleftarrow C_1(X) \longleftarrow$$

where $\varepsilon(\sum_i n_i\sigma_i) \coloneqq \sum_i n_i$ is the augmentation homomorphism from before. Define the *reduced homology groups* of X by $\widetilde{H}_\bullet(X) \coloneqq H_\bullet(\widetilde{C}_\bullet(X))$.

Using the following remark, we can easily transfer from homology and reduced homology when convenient.

Remark 2.1.4.1. If $H_0(X) \to \mathbb{Z}$ is the map induced by ε on homology, one can easily show

$$\widetilde{H}_k(X) = \begin{cases} \ker(H_0(X) \xrightarrow{\varepsilon} \mathbb{Z}) & \text{if } k = 0, \\ H_k(X) & \text{if } k \geq 1. \end{cases}$$

2.1.5. **Relative homology.** Computations are easy when we form exact sequences, and one naturally does this by extending "easy" maps to sequences on the left or right. Such a map is an inclusion $A \hookrightarrow X$, which naturally leads us to topological pairs. Fix a topological pair (X, A), so that $X \in \mathsf{Top}$ and $A \subseteq X$. Consider the quotient groups $C_q(X, A) \coloneqq C_q(X)/C_q(A)$. Define boundary maps $\partial_q\colon C_q(X, A) \to C_{q-1}(X, A)$ as in the following commutative diagram.

$$\begin{tikzcd}
C_q(X) \arrow[rr, "\partial_q^X"] \arrow[d, "Q_q"] & & C_{q-1}(X) \arrow[d, "Q_{q-1}"] \\
C_q(X)/C_q(A) \arrow[rru, dashed, "\overline{\partial}_q^X"] \arrow[rr, "\partial_q \coloneqq Q_{q-1} \circ \overline{\partial}_q^X"'] & & C_{q-1}(X)/C_{q-1}(A)
\end{tikzcd}$$

This gives us a chain complex $C_\bullet(X, A)$, and we define the homology of the topological pair (X, A) by $H_q(X, A) \coloneqq H_q(C_\bullet(X, A))$.

The following result is a quick corollary of a result from homological algebra that a SES of chain complexes induces a LES in homology.

Proposition 2.1.5.1. *There is a long exact sequence of groups*

$$\cdots \longrightarrow H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial} H_{q-1}(A) \longrightarrow \cdots$$

Proof. We have a SES $0 \to C_q(A) \to C_q(X) \to C_q(X)/C_q(A) \to 0$ at each level q and chain maps $\partial_q^X|_A, \partial_q^X$, and $\overline{\partial}_q^X$, to give us the SES of chain complexes $0 \to C_\bullet(X) \to C_\bullet(X) \to C_\bullet(X)/C_\bullet(A) \to 0$. The result then follows from the algebraic result that a SES of chain complexes induces a LES in homology. □

2.1.6. **Eilenberg–Steenrod axioms.** Singular homology follows the following properties, which we will prove later. For now, we will state and use them to display their usefulness as tools.

Theorem 2.1.6.1 (Homotopy). *If $f, g\colon X \to Y$ are homotopic then the induced maps $f_*, g_*\colon H_k(X) \to H_k(Y)$ on homology coincide.*

Theorem 2.1.6.2 (Excision). *Let (X, A) be a pair and let Z be a subset of A. If $\overline{Z} \subseteq A^\circ$, then the inclusion of pairs $i\colon (X \smallsetminus Z, A \smallsetminus Z) \hookrightarrow (X, A)$ induces an isomorphism on homology $i_*\colon H_q(X \smallsetminus Z, A \smallsetminus Z) \xrightarrow{\cong} H_q(X, A)$.*

Theorem 2.1.6.3 (Functoriality). *H_k is a functor $\mathsf{Top} \to \mathbb{Z}Mod$ given by $X \mapsto H_k(X)$ and $(f\colon X \to Y) \mapsto f_*\colon H_k(X) \to H_k(Y))$, where f_* is the map induced by the chain map $f_\sharp\colon C_\bullet(X) \to C_\bullet(Y)$ that sends the k-simplex $\sigma\colon \Delta^k \to X$ to the k-simplex $f_\sharp\sigma \coloneqq f\circ\sigma\colon \Delta^k \to Y$ on Y.*

Theorem 2.1.6.4 (Additivity). *H_k is an additive functor, i.e., if X has path-connected components X_α and inclusions $\iota^\alpha\colon X_\alpha \hookrightarrow X$, then the map $\bigoplus_\alpha \iota^\alpha_*\colon \bigoplus_\alpha H_k(X_\alpha) \to H_k(X)$ is an isomorphism.*

Theorem 2.1.6.5 (Mayer–Vietoris exact sequence). *If $X = U \cup V$ for open subsets U, V of X with inclusion maps $i^U\colon U \cap V \hookrightarrow U$, $j^U\colon U \hookrightarrow X$, $i^V\colon U \cap V \hookrightarrow V$, and $j^V\colon V \hookrightarrow X$, then there is a long exact sequence*

$$\cdots \longrightarrow H_k(U \cap V) \xrightarrow{i^U_* \oplus i^V_*} H_k(U) \oplus H_k(V) \xrightarrow{i^U_* - j^V_*} H_k(X) \xrightarrow{\partial_k} H_{k-1}(U \cap V) \longrightarrow \cdots$$

and ∂ is natural in the sense that for any map $f\colon X \to X'$, the following diagram commutes.

$$\begin{tikzcd}
H_k(X) \arrow[r, "\partial^X"] \arrow[d, "f_*"'] & H_{k-1}(U \cap V) \arrow[d, "f_*"] \\
H_k(X') \arrow[r, "\partial^{X'}"] & H_{k-1}(U' \cap V')
\end{tikzcd}$$

Theorem 2.1.6.6 (Dimension). *For the point space $\mathrm{pt} \in \mathsf{Top}$, $H_k(\mathrm{pt}) = \delta_{k=0}\mathbb{Z}$.*

2.1.7. **Example computations.** Let's use the Eilenberg–Steenrod Axioms now to do calculations.

Example 2.1.7.1. Here we compute $H_q(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{0\})$. We have the long exact sequence

$$\longrightarrow \cancelto{0}{\widetilde{H}_q(\mathbb{R}^n)} \longrightarrow \widetilde{H}_q(\mathbb{R}^n, \mathbb{R}^n \smallsetminus 0) \longrightarrow \cancelto{\widetilde{H}_{q-1}(S^{n-1})}{\widetilde{H}_{q-1}(\mathbb{R}^n \smallsetminus 0)} \longrightarrow \cancelto{0}{\widetilde{H}_{q-1}(\mathbb{R}^n)} \longrightarrow$$

Hence $\widetilde{H}_q(\mathbb{R}^n, \mathbb{R}^n \smallsetminus 0) = \delta_{q=n}\mathbb{Z}$. (Note that this also means $H_q(\mathbb{R}^n, \mathbb{R}^n \smallsetminus 0) = \delta_{q=n}\mathbb{Z}$, since for pairs the reduced homology coincides with homology.)

Theorem 2.1.7.2 (Homology of S^n). *We have $\widetilde{H}_k(S^n) = \delta_{k=n}\mathbb{Z}$.*

Proof. We proceed by induction over n. If $n = 0$ then $S^- = \{-1\} \amalg \{1\}$, so $H_k(S^0) \cong H_k(\mathrm{pt}) \oplus H_k(\mathrm{pt}) \cong \delta_{k,0}\mathbb{Z} \oplus \delta_{k,0}\mathbb{Z}$, where the first isomorphism is by additivity of H_k and the second is by dimensionality. This is 0 if $k \neq 0$ and $\mathbb{Z} \oplus \mathbb{Z}$ if $k = 0$. Thus $\widetilde{H}_k S^0 \cong \mathbb{Z}$, affirming the claim. Now suppose the claim holds for S^{n-1} for some $n \geq 1$. The idea is to apply the Mayer–Vietoris sequence to S^n. Let $U = S^n \smallsetminus n$ and $V = S^n \smallsetminus s$ for n (resp. s) the north (resp. south) poles of S^n. Then $U \cap V = S^n \smallsetminus \{u, v\}$. Note that we have an inclusion S^{n-1} into $U \cap V$. We can choose open sets and then use homotopy equivalence to get

$$\longrightarrow \cancelto{0}{\widetilde{H}_k(U)} \oplus \cancelto{0}{\widetilde{H}_k(V)} \longrightarrow \widetilde{H}_k(S^n) \xrightarrow{\partial} \cancelto{\cong \widetilde{H}_{k-1}(U\cap V)}{\widetilde{H}_{k-1}(S^{n-1})} \longrightarrow \cancelto{0}{\widetilde{H}_{k-1}(U)} \oplus \cancelto{0}{\widetilde{H}_{k-1}(V)} \longrightarrow$$

so $0 \to \widetilde{H}_k(S^n) \to \widetilde{H}_{k-1}(S^{n-1}) \to 0$ is exact, so the middle map is an isomorphism. ☐

Lemma 2.1.7.3. *$\widetilde{H}_q(\mathbb{RP}^2) = \delta_{k=0}\mathbb{Z}/(2)$.*

Proof. We can write $\mathbb{RP}^2 = D^2/\{x \sim -x \mid x \in S^1 \subseteq D^2\}$. Now consider $U = (D^2)^\circ$, $V = (D^2 - \{0\})/\sim$, and $U \cap V = (D^2)^\circ - \{0\}$. Then $U \sim \mathrm{pt}$, $U \cap V \sim S^1$ via the map $r^1 \colon (D^2)^\circ - \{0\} \ni z \mapsto z/|z| \in S^1$, and $V \sim S^1$ via the map $r^2 \colon (D^2 - \{0\})/\sim \ni [z] \mapsto (z/|z|)^2 \in S^1$. Now for the interesting part. By the Mayer–Vietoris sequence, we have

$$0 \longrightarrow \widetilde{H}_2(\mathbb{RP}^2) \xrightarrow{\partial} \widetilde{H}_1(U \cap V) \longrightarrow \widetilde{H}_1(U) \oplus \widetilde{H}_1(V) \longrightarrow \widetilde{H}_1(\mathbb{RP}^2) \xrightarrow{\partial} 0$$

The leftmost term vanishes. Observe then $\widetilde{H}_q(\mathbb{RP}^2) = 0$ for $q \neq 1, 2$ and $\widetilde{H}_2(\mathbb{RP}^2) = \ker i_*$, where we relabel our diagram as

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{H}_2(\mathbb{RP}^2) \arrow[r, hook, "\partial"] & \widetilde{H}_1(U \cap V) \arrow[r, "i_*^V"] & \cancelto{0}{\widetilde{H}_1(U)} \oplus \widetilde{H}_1(V) \arrow[r, two heads] & \widetilde{H}_1(\mathbb{RP}^2) \arrow[r, "\partial"] & 0
\end{tikzcd}$$

Thus $\widetilde{H}_2(\mathbb{RP}^2) \cong \ker i_*^V$. $\widetilde{H}_1(\mathbb{RP}^2) \cong \widetilde{H}_1(V)/\operatorname{im} i_*^V = \operatorname{coker} i_*^V$.

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{H}_2(\mathbb{RP}^2) \arrow[r, hook, "\partial"] & \widetilde{H}_1(U \cap V) \arrow[r, "i_*^V"] \arrow[d, "r_*^1"', "\cong"] & \cancelto{0}{\widetilde{H}_1(U)} & \oplus & \widetilde{H}_2(V) \arrow[r, two heads] \arrow[d, "r_*^2"', "\cong"] & \widetilde{H}_1(\mathbb{RP}^2) \arrow[r, "\partial"] & 0 \\
 & & \widetilde{H}_1(S^1) \arrow[rrr, "f_*"] & & & \widetilde{H}_1(S^1) & & \\
 & & \pi_1(S^1) = \pi_1(S^1)^{\mathrm{ab}} \arrow[u, "h"', "\cong"] \arrow[rrr, "f_* = \times 2"] \arrow[d, dash, "\cong"] & & & \pi_1(S^1) \cong \pi_1(S^1)^{\mathrm{ab}} \arrow[u, "h"', "\cong"] \arrow[d, "\cong"] & & \\
 & & \mathbb{Z} \arrow[rrr, "1 \mapsto 2"] & & & \mathbb{Z} & &
\end{tikzcd}$$

Hence $\widetilde{H}_1(\mathbb{RP}^2) \cong \widetilde{H}_1(V)/\operatorname{im} i_*^V \cong \mathbb{Z}/(2)$. ☐

Example 2.1.7.4. Let $x_1, \ldots, x_l$ be distinct points of $\mathbb{R}^n$. Calculate the reduced homology groups of the space $\mathbb{R}^n \smallsetminus \{x_1, \ldots, x_l\}$. [Hint: Compare the homology groups of

$\mathbb{R}^n \setminus \{x_1, \ldots, x_l\}$ with those of $\mathbb{R}^n$ via the Mayer–Vietoris sequence.]

Proof. Instead of decomposing $\mathbb{R}^n \setminus \{x_1, \ldots, x_l\}$, we opt to decompose $\mathbb{R}^n$ into a union involving $\mathbb{R}^n \setminus \{x_1, \ldots, x_l\}$. Let $U = B_1 \cup \cdots \cup B_l$ be a union of disjoint open disks $B_1, \ldots, B_l$ centered at $x_1, \ldots, x_l$, respectively. Let $V = \mathbb{R}^n \setminus \{x_1, \ldots, x_l\}$. Then U and V are open subsets of $U \cup V = \mathbb{R}^n$. We observe the following facts.

- U is homotopic to the disjoint union of closed unit n-balls $\coprod_{i=1}^{l} D^n$, which has
$$H_k(U) = H_k\left(\coprod_{i=1}^{l} D^n\right) \cong \bigoplus_{i=1}^{l} H_k(D^n) \cong \bigoplus_{i=1}^{l} H_k(\mathrm{pt}) \cong \delta_{k,0}\mathbb{Z}^l.$$
- Denote by B_i^* the punctured ball $B_i \setminus \{x_i\}$. Then $U \cap V = B_1^* \amalg \cdots \amalg B_l^*$ is homotopic to $S^{n-1} \amalg \cdots \amalg S^{n-1}$. Thus
$$H_k(U \cap V) = H_k\left(\coprod_{i=1}^{l} S^{n-1}\right) \cong \bigoplus_{i=1}^{l} H_k(S^{n-1}) \cong (\delta_{k,0} + \delta_{k,n-1})\mathbb{Z}^l.$$

Then the Mayer–Vietoris exact sequence contains the following.

$$\begin{tikzcd}
H_{k-1}(\mathbb{R}^n) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & H_k(U \cap V) \arrow[r, "i_*^U \oplus i_*^V"] \arrow[d, phantom, sloped, "\cong"] & H_k(U) \oplus H_k(V) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & H_k(\mathbb{R}^n) \arrow[d, phantom, sloped, "\cong"] \\
\delta_{k-1,0}\mathbb{Z} & (\delta_{k,0} + \delta_{k,n-1})\mathbb{Z}^l & \delta_{k,0}\mathbb{Z}^l \oplus H_k(V) & \delta_{k,0}\mathbb{Z}
\end{tikzcd}$$

If $k \neq 0$, then this sequence becomes $0 \to \delta_{k=n-1}\mathbb{Z}^l \to H_k(V) \to 0$. Thus for $k \geq 1$ we have $H_k(V) \cong \delta_{k,n-1}\mathbb{Z}^l$. Finally, since V has exactly one connected component (as the set of points $\{x_1, \ldots, x_l\}$ is finite), we have $\widetilde{H}_0(V) = 0$, and thus $\widetilde{H}_0(V) = 0$. Since homology groups agree with reduced homology groups $k \geq 1$, $\widetilde{H}_k(V) \cong \delta_{k,n-1}\mathbb{Z}^l$ for $k \geq 1$. We conclude $\widetilde{H}_k(\mathbb{R}^n \setminus \{x_1, \ldots, x_l\}) \cong \delta_{k=n-1}\mathbb{Z}^l$. □

2.2. Some applications of homology

2.2.1. **Complements of disks and spheres in spheres.** A *k-disk* (resp. *k-sphere*) in S^n is a subset of S^n homeomorphic to D^k (resp. S^k). here we make observations about the homology groups of the complement and a k-disk D_k in S^n or of a k-sphere S_k in S^n.

Proposition 2.2.1.1 (Homology of the complement of k-disk in S^n)**.** *If D_k is a k-disk in S^n, then $\widetilde{H}_q(D_k^c) = 0$.*

Proof. We argue by induction over k, with $k = 0$ being trivial. For the induction step, we have for $I = [0, 1]$ that $I^k \cong D^k \overset{h}{\hookrightarrow} S^n$, $\operatorname{im} h = k$-disk D_k. We can decompose I^k as $([0, 1/2] \cup [1/2, 1]) \times I^{k-1}$ and distribute to give us an upper cube $C_{k,+}$ and a lower cube $C_{k,-}$ so that we can think of $D_k \cong I^k$ as a union of upper and lower cubes, i.e. $D_k = C_{k,+} \cup C_{k,-}$.

Notice also that $D_{k-1} \coloneqq C_{k,+} \cap C_{k,-} = h(\{1/2\} \times I^{k-1})$ is the $(k-1)$ disk in S^n. The Mayer-Vietoris sequence gives that the sequence

$$\widetilde{H}_{q+1}(D_{k-1}) \xrightarrow{\partial} \widetilde{H}_q(D_k^c) \longrightarrow \widetilde{H}_q(C_{k,+}) \oplus \widetilde{H}_q(C_{k,-}^c) \longrightarrow \widetilde{H}_q(D_{k-1}^c)$$

is exact. The rightmost term is zero and the leftmost term is 0 by the induction hypothesis, so the map $\widetilde{H}_q(D_k^c) \to \widetilde{H}_q(C_+^c) \oplus \widetilde{H}_q(C_-^c)$ is an isomorphism. □

Theorem 2.2.1.2 (Homology of the complement of k-sphere in S^n)**.** *If S_k is a k-sphere in S_1^n then $\widetilde{H}_q(S_k^c) \cong \widetilde{H}_q(S^{n-k-1}) = \delta_{q=n-k-1}\mathbb{Z}$.*

Proof. We argue by induction over k. For the base case $k = 0$, note $S^0 \hookrightarrow S^n$ implies $S_0^c = \mathbb{R}^n \smallsetminus \mathrm{pt} \sim S^{n-1}$, as desired. For the induction step, write $S^n \supseteq S_k = D_{k,+} \cup D_{k,-}$, the union of the hemispheres of the disk. Then the intersection $D_{k,+} \cap D_{k,-} =: S_{k-1}$ is a $(k-1)$-sphere in S^n. So $S_k^c = (D_{k,+} \cup D_{k,-})^c = D_{k,+}^c \cap D_{k,-}^c$, $S_{k-1}^c = (D_{k,+} \cap D_{k,-})^c = D_{k,+}^c \cup D_{k,-}^c$.

We now apply the Mayer Vietoris sequence. By the examples from class we can intuitively believe the statement of the proposition, that is, that $\widetilde{H}_q(D_{k,\pm}^c) = 0$ and $\widetilde{H}_q(S_{k-1}^c) = \delta_{q=n-(k-1)-1}\mathbb{Z} = \delta_{q=n-k}\mathbb{Z}$, so we have an exact sequence

$$0 \longrightarrow \widetilde{H}_{n-k}(S_{k-1}^c) \xrightarrow{\partial} \widetilde{H}_{n-k-1}(S_k^c) \longrightarrow \widetilde{H}_{n-k-1}(D_{k,+}^c) \oplus \widetilde{H}_{n-k-1}(D_{k,-}^c) = 0 \oplus 0$$

Thus ∂ is an isomorphism. □

Corollary 2.2.1.3 (Jordan curve theorem)**.** *If $S_1 \subseteq S^2$ is a Jordan curve, then the complement S_1^c has two components.*

2.2.2. **Local homology.** Let X be Hausdorff. Given $x_0 \in X$, the map

$$H_q(U, U \smallsetminus x_0) \xrightarrow{\cong} H_q(X, X \smallsetminus x_0)$$

induced by the inclusion is an isomorphism. This follows from setting $Z = X \smallsetminus U = U^c$, the complement of U in X, and excising Z from X. Indeed, then $(X \smallsetminus Z, (X \smallsetminus x_0) \smallsetminus Z) = (U, U \smallsetminus x_0)$, so the map induced by the inclusion, $i_*\colon H_q(X \smallsetminus Z, A \smallsetminus Z) \xrightarrow{\cong} H_q(X, A)$, is an isomorphism by the excision theorem of homology. U was an arbitrary open subset of X, so $H_q(X, X \smallsetminus x_0) \cong H_q(X, X \smallsetminus x_0)$ for all open subsets U of X. Due to the isomorphism property of excision, $H_q(X, X \smallsetminus x_0)$ depends *only* on a neighborhood of x_0. This motivates the following terminology. $H_q(X, X \smallsetminus x_0)$ is called the *local homology group* of X at x_0.

Proposition 2.2.2.1 (Application to manifold invariants)**.** *Let M, N be manifolds and consider a homeomorphism $f\colon M \xrightarrow{\cong} N$. Using local homology, we can show that the following hold.*

(i) $\dim M = \dim N$.
(ii) $f(\partial M) \subseteq \partial N$

(iii) *The induced map* $f\colon \partial M \to \partial N$ *is a homeomorphism.*

Proof. Since M is locally homeomorphic to $\mathbb{R}^m$ (if $\partial M = \varnothing$), we know by the previous example that $H_q(M, M \smallsetminus x) = \delta_{q,m}\mathbb{Z}$, where $m = \dim M$. Since this determines the dimension, we conclude point (i).

For (ii), take an open chart $(U, \varphi\colon U \to \mathbb{R}^m_{\geq 0})$ sending $U \ni x$ to either (case I) $\varphi(x) \in \mathbb{R}^m_{\geq 0}$ or (case II) $\varphi(x) \in \partial\mathbb{R}^m_{\geq 0}$. In case I, $H_q(M, M \smallsetminus x) \cong H_q((\mathbb{R}^m_{\geq 0})^\circ, (\mathbb{R}^m_{\geq 0})^m)^\circ \smallsetminus y) = \delta_{q,m}\mathbb{Z}$. In case II, $H_q(M, M \smallsetminus x) \cong H_q(\mathbb{R}^m_{\geq 0}, \mathbb{R}^m_{\geq 0} \smallsetminus y)$, where y is a boundary point. $\mathbb{R}^m_{\geq 0}$ is contractible (since any convex subspace is contractible). In fact, $\mathbb{R}^m_{\geq 0} \smallsetminus y$, where y is a boundary point, is also contractible. Indeed, $\mathbb{R}^m_{\geq 0} \smallsetminus y$ is *star shaped*, that is, there exists a point that is connected to any other point via a line. Then by the long exact sequence of pairs, we conclude $0 \cong H_q(\mathbb{R}^m_{\geq 0}, \mathbb{R}^m_{\geq 0} \smallsetminus y)$. This proves (ii).

For (iii), note f maps ∂M to ∂N. The homeomorphism has inverse, and by the same argument you map the boundary of one to the other and the other is the other to the first one, so they must be mutual inverses. ☐

2.2.3. **Degree of a map.** Recall a continuous map $f\colon S^n \to S^n$ induces a group homomorphism $f_*\colon \widetilde{H}_n(S^n) \to \widetilde{H}_n(S^n)$ between homologies. If α is the generator for $H_n(S^n) \cong \mathbb{Z}$, then the *degree* of f is the unique integer $\deg(f)$ satisfying $f_*(\alpha) = \deg(f)\cdot\alpha$. The degree of a map $f\colon S^1 \to S^1$ is called the *winding number* of f.

Proposition 2.2.3.1 (Properties of the degree)**.** *Let* $f, g\colon S^n \to S^n$ *be continuous maps.*

(i) *If* f *is homotopic to* g *then* $\deg f = \deg g$. *(In fact, the converse to (i) also holds, but we will not prove this here.)*

(ii) $\deg(f \circ g) = \deg(f)\deg(g)$.

(iii) *If* f *is not surjective then* $\deg(f) = 0$.

(iv) *The degree of the reflection map* $r\colon S^n \to S^n$ *sending* $(x_0, \dots, x_n) \mapsto (-x_0, x_1, \dots, x_n)$ *is* $\deg(r) = -1$.

Proof. (i) This is an immediate application of homotopy invariance of homology.

(ii) This follows from the fact the diagram

$$\begin{tikzcd}
\widetilde{H}_n(S^n) \arrow[r, "g_*"] \arrow[rr, bend left, "(f\circ g)_*"] & \widetilde{H}_n(S^n) \arrow[r, "f_*"] & \widetilde{H}_n(S^n) \\
\alpha \arrow[r, mapsto] & \deg(f)\alpha \arrow[r, mapsto] & \deg(f)\deg(g)\alpha
\end{tikzcd}$$

commutes.

(iii) If $x \in S^n$ is not in the image of f, then we can factor f as

$$\begin{tikzcd} S^n \arrow[r, "f|_{S^n \smallsetminus x}"] \arrow[rr, bend left, "f"] & S^n \smallsetminus x \arrow[r, hook, "i"] & S^n \end{tikzcd}$$

Applying homology, we get the commutative diagram

$$\begin{tikzcd} H_n(S^n) \arrow[r, "(f|_{S^n \smallsetminus x})_*"] \arrow[rr, bend left, "f_*"] & H_n(S^n \smallsetminus x) \arrow[r, hook, "i_*"] & H_n(S^n) \end{tikzcd}$$

Hence $f_* = 0$, since $H_n(S^n \smallsetminus x) = 0$.

(iv) We leave this to the reader. □

Theorem 2.2.3.2. *S^n has a nowhere vanishing vector field if and only if n is odd.*

Proof. The reverse implication involves construction of a vector field. When $n = 2k+1$ we have $S^{2k+1} \subseteq \mathbb{C}^{2k}$, so we can consider the vector field $S^{2k+1} \to TS^{2k+1}$ valued in the tangent space given by sending $(z_0, \dots, z_x)$ to $i \cdot (z_0, \dots, z_k) = (iz_0, \dots, iz_k)$. This is a non-vanishing vector field.

Conversely, suppose there is a nowhere vanishing vector field $S^n \to TS^n$ sending $x \mapsto v(x) \perp x \in \mathbb{R}^{n+1}$. Without loss of generality take $\|v(x)\| = 1$ for all x. Now make a homotopy $F_\bullet \colon S^n \times I \to S^n$ sending $(x,t) \mapsto \cos(\pi t) \cdot x + \sin(\pi t) \cdot v(x)$, a rotation. Now $F_0 = \mathrm{id}_{S^n}$, so it follows from homotopy invariance of the degree that $F_1 = -\mathrm{id}_{S^n}$, so $\deg(F_0) = \deg(F_1) = 1$ and thus n is odd. □

Example 2.2.3.3. Recall the orthogonal group $O_n :=$ {linear isometries $f \colon \mathbb{R}^n \to \mathbb{R}^n$}. If $f \in O_n$, then $\deg(f) \in \{\pm 1\}$. Indeed, the orthogonal group O_n has two connected components (since $\det \colon O_n \to \{\pm 1\}$ is continuous and hence must have two components, constant on each component.)

Exercise 2.2.3.4 (The suspension of a topological space)**.** Let X be any topological space. The (unreduced) *suspension* of X, denoted ΣX, is defined as the quotient space $\Sigma X := (X \times [0,1])/\sim$, where the equivalence relation $\sim$ is generated by $(x,0) \sim (x',0)$ and $(x,1) \sim (x',1)$ for all $x, x' \in X$.

1. Show that ΣS^n is homeomorphic to S^{n+1}.

2. Construct an isomorphism $H_{q+1}(\Sigma X) \xrightarrow{\cong} H_q(X)$. This isomorphism is called the *suspension isomorphism*. Hint: Think of the suspension isomorphism as a generalization of the isomorphism $\widetilde{H}_{q+1}(S^{n+1}) \cong \widetilde{H}_q(S^n)$ proved in class. That proof used the decomposition of S^{n+1} as a union of $U = S^{n+1} \smallsetminus$ north pole and $V = S^{n+1} \smallsetminus$ south

pole. In this more general situation, the subspaces $U = \{[x,t] \in \Sigma \mid 0 \leq t < 1\}$ and $V = \{[x,t] \in \Sigma \mid 0 < t \leq 1\}$ of ΣX play an analogous role.

Exercise 2.2.3.5 (Examples of functoriality of suspension)**.**

1. It can be shown that the suspension ΣS^n is homeomorphic to S^{n+1}. Show that for a map $f\colon S^n \to S^n$, the degree of the map $S^{n+1} = \Sigma S^n \xrightarrow{\Sigma f} \Sigma S^n = S^{n+1}$ is equal to the degree of f.

2. Let $r_n\colon S^n \to S^n$ be the reflection map $(x_0, \ldots, x_n) \mapsto (-x_0, \ldots, x_n)$. Show that $\deg(r_n) = -1$. Hint: Using part (a), proceed by induction over n, starting at $n = 0$.

3. $\deg(\text{antipodal map } S^n \to S^n) = (-1)^{n+1}$.

2.2.4. **Local degree and the degree theorem.** Let $x \in S^n$ and let $f\colon U \to S^n$ be a map defined on an open neighborhood U of x. If the preimage $f^{-1}(y)$ consists only of the point x, then we can think of f as a map of pairs $f\colon (U, U \smallsetminus x) \to (S^n, S^n \smallsetminus y)$ where $y = f(x)$. So let $f\colon U \to S^n$ be a map satisfying the assumption above, possibly after replacing U by a smaller neighborhood of $x \in S^n$. Let $f_*\colon H_n(U, U \smallsetminus x) \longrightarrow H_n(S^n, S^n \smallsetminus y)$ be the induced map of local homology groups. We define the *local degree* of f at x, denoted $\deg(f,x)$, is the unique integer such that the image of α under the morphisms in the diagram below is $\deg(f,x) \cdot \alpha$, i.e., $f_*(\alpha) = \deg(f,x) \cdot \alpha$. Abusing notation here we denote by α the generator of $H_n(U, U \smallsetminus x)$ as well as the generator of $H_n(S^n, S^n \smallsetminus y)$ that correspond to our chosen generator $\alpha \in \widetilde{H}_n(S^n)$ via the isomorphisms

$$\begin{tikzcd}
 & & H_q(U, U \smallsetminus x) \arrow[r, "f_*"] \arrow[d, "\text{excision}"', "\cong"] & H_q(S^n, S^n \smallsetminus y) & & \\
 & & H_q(S^n, S^n \smallsetminus y) & & & \\
\alpha & \in & \widetilde{H}_q(S^n) \arrow[u, "\text{LES from the pair } (S^n, S^n \smallsetminus x)", "\cong"'] & \widetilde{H}_q(S^n) \arrow[uu, "\cong", "\text{LES from the pair } (S^n, S^n \smallsetminus y)"'] & \ni & \deg(f,x) \cdot \alpha
\end{tikzcd}$$

Theorem 2.2.4.1 (Degree theorem)**.** *If the fiber of a point $y \in S^n$ of a continuous map $f\colon S^n \to S^n$ is a finite set $\{x_1, \ldots, x_m\}$, then* $\deg(f) = \sum_{i=1}^{m} \deg(f, x_i)$.

Proof. To define the local degree $\deg(f, x_i) \in \mathbb{Z}$, consider the commutative diagram

$$\begin{tikzcd}
\widetilde{H}_n(S^n) \arrow[r, "f_*"] \arrow[d] & \widetilde{H}_n(S^n) \arrow[d, "\cong"', "\substack{\text{LES of pairs \&} \\ S^n \setminus y \text{ is contractible}}"] \\
H_n(S^n, S^n \setminus \{x_1, \dots, x_n\}) \arrow[r, "f_*"] & H_n(S^n, S^n \setminus y) \arrow[d, equal] \\
H_n(\coprod_{i=1}^m (U_i, U_i \setminus x_i)) \arrow[r, "f_*"] \arrow[u, "\text{excision}", "\cong"'] & H_n(S^n, S^n \setminus y) \arrow[d, equal] \\
\bigoplus_{i=1}^m H_n(U_i, U_i \setminus x_i) \arrow[r, "\oplus_{i=1}^m (f|_{U_i})_*"] \arrow[u, "\substack{\text{LES of pairs} \\ \text{\& contractible}}", "\cong"'] & H_n(S^n, S^n \setminus y)
\end{tikzcd}$$

where any unlabeled map is induced by the inclusion maps. Sending a generator α for $\widetilde{H}_n(S^n)$ through the isomorphisms, we obtain generators β_i for the local homology groups $H_n(U_i, U_i \setminus x_i)$. The top horizontal map encodes the degree of f in the sense that the generator β_i for $H_n(U_i, U_i \setminus x_i)$ is sent to

$$(f|_{U_i})_*(\beta_i) = \deg(f, x_i)\beta_i = \left(\sum_{i=1}^m \deg(f, x_i) \right) \beta_i.$$

It then remains to show that the composition of the vertical maps on the left map the generator α for $\widetilde{H}_n(S^n)$ to the diagonal element $(\alpha, \dots, \alpha)$. To show this, suppose that the image of α under the vertical maps is $(\alpha_1, \dots, \alpha_m) \in \bigoplus_{i=1}^m H_n(U_i, U_i \setminus x_i)$. To show that $\alpha_i = \alpha \in H_n(U_i, U_i \setminus x_i)$ we look at the following diagram.

$$\begin{tikzcd}
 & \widetilde{H}_n(S^n) \arrow[dl, "\cong"'] \arrow[d] \\
H_n(S^n, S^n \setminus x_i) & H_n(S^n, S^n \setminus \{x_1, \dots, x_n\}) \arrow[l] \\
 & H_n(U_i, U_i \setminus x_i) \arrow[ul, "\cong"] \arrow[u]
\end{tikzcd}$$

Its commutativity implies $\alpha_i = \alpha$. □

Example 2.2.4.2. We have the inclusion $O_n = \{\text{linear isometries } \mathbb{R}^n \xrightarrow{f} \mathbb{R}^n\} \hookrightarrow \mathrm{GL}_n$. On the other hand, there is a map in the other direction by applying Gram–Schmidt algorithm to the rows of the invertible matrix. In fact, the Gram–Schmidt algorithm shows that O_n is a deformation retract of GL_n.

$$\begin{tikzcd}
{[f]} \arrow[d, mapsto] & \in & \pi_0 O_n \arrow[d] & = & \substack{\text{set of connected} \\ \text{components of } O_n} & {[f]} \arrow[d, mapsto] & \in & \pi_0 \mathrm{GL}_n \arrow[d] \\
\det(f) & \in & \{\pm 1\} & & & \mathrm{sign}(f) & \in & \in \{\pm 1\}
\end{tikzcd}$$

Lemma 2.2.4.3 (Local Degree of Linear Maps)**.**

(a) *If* $f\colon \mathbb{R}^n \overset{\cong}{\to} \mathbb{R}^n$ *is a linear isomorphism then* $\deg(f, 0) = \operatorname{sign}\det(f) \in \{\pm 1\}$*.*

(b) *If* $f\colon \mathbb{R}^n \to \mathbb{R}^n$ *continuous and differentiable at* $x_0 \in \mathbb{R}^n$*, and* δf_{x_0} *is invertible, then* $\det(f, x_0) = \operatorname{sign}\det(\delta f_{x_0})$*. In other words,*

$$\deg(f, x_0) = \begin{cases} +1 & \textit{if } \delta f_{x_0} \textit{ is orientation-preserving,} \\ -1 & \textit{if } \delta f_{x_0} \textit{ is orientation-reversing.} \end{cases}$$

2.2.5. **Orientations and homological orientations.** We now turn to a discussion of orientations, since that fits well in this context. Let V be a real vector space of dimension n. Then the set

$$\mathrm{Or}(V) \coloneqq \{\text{vector space orientations of } V\}.$$

consists of two elements. We now introduce a similar notion in terms of homology groups.

Definition 2.2.5.1 (Homological orientation)**.** Define the *homological orientations* of a vector space V by

$$\mathrm{HOr}(V) \coloneqq \{\text{generators of } H_n(V, V \smallsetminus 0) \cong \mathbb{Z}\}.$$

Then $\mathrm{HOr}(V)$ consists of two elements since $\mathbb{Z}$ has two generators as a $\mathbb{Z}$-module, namely 1 and -1.

Lemma 2.2.5.2. *If* V *is an* n*-dimensional real vector space, then there is a natural bijection* $\mathrm{Or}(V) \overset{B_V}{\longleftrightarrow} \mathrm{HOr}(V)$ *between vector space orientations and homological orientations.*

In other words, if $g\colon V \overset{\cong}{\to} W$ *is an isomorphism of vector spaces, then the diagram*

$$\begin{tikzcd}
\lbrack\mathbb{R}^n \xrightarrow[\cong]{f} V\rbrack \arrow[d, mapsto] & \in & \mathrm{Or}(V) \arrow[r, leftrightarrow, "B_V"] \arrow[d, "g_*"] & \mathrm{HOr}(V) \arrow[d, "g_*"] & \ni & \text{generator } \alpha \text{ of } H_n(V, V \smallsetminus 0) \arrow[d, "\cong"'] \\
\lbrack\mathbb{R}^n \xrightarrow{f} V \xrightarrow{g} W\rbrack & \in & \mathrm{Or}(W) \arrow[r, "B_W"] & \mathrm{HOr}(W) & \ni & \text{generator } g_*(\alpha) \text{ of } H_n(W, W \smallsetminus 0)
\end{tikzcd}$$

commutes.

Proof. Fix a generator $\alpha \in H_n(\mathbb{R}^n \smallsetminus \mathbb{R}^n, \mathbb{R}^n \smallsetminus 0)$. Then

$$\begin{tikzcd}
B_V\colon & O_V(V) \arrow[r] & \mathrm{HOr}(V) & H_n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus 0) \arrow[r, mapsto] & H_n(V, V \smallsetminus 0) \\
& \lbrack\mathbb{R}^n \xrightarrow[\cong]{f} V\rbrack \arrow[r, mapsto] & f_*(\alpha) & \alpha \arrow[r, mapsto] & f_*(\alpha)
\end{tikzcd}$$

□

Example 2.2.5.3.

(i) $f_m\colon S^1 \to S^1$, $z \mapsto z^m$. Pick $y = 1 \in S^1$, $f^{-1}(1) = \{m\text{th roots of unity}\}$. Then δf_{x_0} is orientation-preserving, so $\deg(f, x_0) = +1 \implies \deg(f)$ is the number of elements in $f^{-1}(1)$, which is m for $k > 0$.

(ii) $f\colon S^n \to S^n$ a differentiable map, $y \in S^n$ a regular value (that is, for each point x in the finite preimage $f^{-1}(y)$, which we note has dimension zero, the differential $\mathrm{d}f_x\colon \mathbb{R}^n \to \mathbb{R}^n$ is surjective (and hence an isomorphism). Then by the inverse function theorem, f is a local diffeomorphism in an open neighborhood of $x \in f^{-1}(y)$. Then $\deg(f) = \sum_{x \in f^{-1}(y)} \varepsilon(x)$, where

$$\varepsilon(x) = \begin{cases} +1 & \text{if } \mathrm{d}f_x \text{ is orientation-preserving,} \\ -1 & \text{otherwise.} \end{cases}$$

2.2.6. **Good pairs.** A topological pair (X, A) is called a *good pair* if there exists a nonempty open neighborhood V of A in X that deformation retracts to A.

Example 2.2.6.1.

(i) The pair (D^n, S^{n-1}) is a good pair for each n, since $U \coloneqq D^n \smallsetminus \{0\}$ deformation retracts onto S^{n-1}.

(ii) For any space X and a map $f\colon S^{n-1} \to X$, by taking the complement of 0 again as above, we see $(X \cup_f D^n, X)$ is a good pair.

(iii) The previous item shows that for any CW complex X, the pair (X^n, X^{n-1}) is a good pair.

Proposition 2.2.6.2. *If (X, A) is a good pair, then the quotient map $q\colon (X, A) \twoheadrightarrow (X/A, A/A)$ induces an isomorphism $q_*\colon H_k(X, A) \xrightarrow{\cong} H_k(X/A, A/A) \cong \widetilde{H}_k(X/A)$, where the rightmost isomorphism follows from a LES induced by triples.*

Proof. We have the commutative diagram

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "\text{LES of } (X,V,A)", "\cong"'] \arrow[d, "q_*"'] & H_n(X,V) \arrow[r, "\text{excision}", "\cong"'] \arrow[d, "q_*"] & H_n(X \smallsetminus A, V \smallsetminus A) \arrow[d, "\cong"', "q_* \text{ (since homeomorphism)}"] \\
H_n(X/A, A/A) \arrow[r, "\text{LES}", "\cong"'] & H_n(X/A, V/A) \arrow[r, "\text{excision}", "\cong"'] & H_n((X/A) \smallsetminus (A/A), (V/A) \smallsetminus (A/A))
\end{tikzcd}$$

□

Let X be a finite-dimensional CW complex. Then (X^n, X^{n-1}) is a good pair, so by the proposition we have $H_k(X^n, X^{n-1}) \cong \widetilde{H}_k(X^n/X^{n-1})$. To see what the quotient space X^n/X^{n-1} looks like, write

$$X^n/X^{n-1} = \left((X^{n-1} \coprod_\alpha D^n_\alpha)/\sim \right) / X^{n-1} = \left(\coprod_\alpha D^n_\alpha \right) \Big/ \left(\coprod_\alpha \partial D^n_\alpha \right).$$

We can contract the denominator to a single point, so X^n/X^{n-1} is homotopic to the wedge sum $\bigvee_\alpha X_\alpha \coloneqq (\coprod_\alpha X_\alpha) \,/\, (\coprod_\alpha \{x_\alpha\})$.

Despite the fact that in general the *reduced* homology functor $\widetilde{H}_k$ is not additive (that is, does not preserve direct sums), the following lemma shows that the reduced homology functor $\widetilde{H}_k$ is additive on wedge sums.

Lemma 2.2.6.3. *If $(X_\alpha, \{x_\alpha\})$ is a good pair and $i_\alpha \colon X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ are the inclusion maps, then $\bigoplus_\alpha (i_\alpha)_* \colon \bigoplus_\alpha \widetilde{H}_k(X_\alpha) \xrightarrow{\cong} \widetilde{H}_k\left(\bigvee_\alpha X_\alpha\right)$.*

Proof. This follows from the diagram

$$\begin{tikzcd}
\bigoplus_\alpha \widetilde{H}_k(X_\alpha) \arrow[r] & \widetilde{H}_k\left(\bigvee_\alpha X_\alpha\right) \\
\bigoplus_\alpha H_k(X_\alpha, x_\alpha) \arrow[u, "\cong"'] \arrow[r, "\cong"] & H_k(\amalg X_\alpha, \amalg\{x_\alpha\}) \arrow[u, "\cong"']
\end{tikzcd}$$

☐

To summarize, X^n/X^{n-1} is the wedge sum of n-spheres, one for each n-cell in X.

Lemma 2.2.6.4. *$H_k(X^n, X^{n-1}) = \delta_{k=n}\mathbb{Z}[\{n\text{-cells } e^n_\alpha\}]$.*

Proof. This follows from the following diagram.

$$\begin{tikzcd}
 & & {} \arrow[d] \arrow[dd, squiggly, shift right=4ex, "\text{LES}"] & {} \arrow[d] \arrow[dd, squiggly, shift left=3ex] & {} \arrow[dd] & & \\
 & & \cancel{H_{n+1}(X^{n+1},X^n)} \rightarrow 0 \arrow[d] & H_{n+1}(X^{n+1},X^n) \arrow[d] & & & \\
H_n(X^0) \arrow[r, "\cong"] & \cdots \arrow[r, "\cong"] & H_n(X^{n-1}) \arrow[r, hook] \arrow[d] \arrow[r, squiggly, shift right=2ex] & H_n(X^n) \arrow[r, two heads] \arrow[d] \arrow[r, squiggly, shift left=2ex] \arrow[dd, squiggly, shift right=3ex] & H_n(X^{n+1}) \arrow[r, "\cong"] \arrow[d] \arrow[dd, squiggly, shift left=3ex, "\text{LES}"] & H_n(X^{n+1}) \arrow[r, "\cong"] & {} \\
 & & {} & H_n(X^n,X^{n-1}) \arrow[d] & \cancel{H_n(X^{n+1},X^n)} \rightarrow 0 \arrow[d] & & \\
 & & & {} & {} & &
\end{tikzcd}$$

☐

Corollary 2.2.6.5. *Let X be a (possibly infinite-dimensional) CW complex.*

(i) *$H_k(X^n) = 0$ for $k > n$.*

(ii) *$H_k(X^n) \to H_k(X)$ is an isomorphism for $k < n$ and surjective for $k = n$.*

2.2.7. **Cellular homology.** The *cellular chain complex* of a CW complex, denoted $C^{\mathrm{CW}}_\bullet$ is defined at degree n by $C^{\mathrm{CW}}_n(X) := H_n(X^n, X^{n-1}) = \bigoplus_\alpha \mathbb{Z}e^n_\alpha$, where the e^n_α are the n-cells of X, and the boundary maps $\delta_n \colon C^{\mathrm{CW}}_n(X) \to C^{\mathrm{CW}}_{n-1}(X)$ are defined in the following diagram, whose bottom row is the LES associated to the triple (X^n, X^{n-1}, X^{n-2}).

$$\begin{tikzcd}
 & & C_n^{\mathrm{CW}}(X) \arrow[r, "\delta_n"] \arrow[d, equal] & C_{n-1}^{\mathrm{CW}}(X) \arrow[d, equal] & & \\
\cdots \arrow[r] & H_n(X, X^{n-2}) \arrow[r] & H_n(X, X^{n-1}) \arrow[r, "\partial =: \delta_n"] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & H_{n-1}(X, X^{n-2}) \arrow[r] & \cdots
\end{tikzcd}$$

Then define the *kth cellular homology group* by $H_k^{\mathrm{CW}} := H_k(C_\bullet^{\mathrm{CW}}(X))$.

Theorem 2.2.7.1. *Cellular homology agrees with singular homology, i.e.,* $H_k^{\mathrm{CW}}(X) \cong H_k(X)$.

Proof. This follows from the following diagram, whose top-left to bottom-right sequence is the LES for the pair (X^n, X^{n-1}) and whose top and bottom bottom-left to top-right sequence are the LESs for the pairs (X^{n+1}, X^n) and (X^{n-1}, X^{n-2}) respectively.

$$\begin{tikzcd}
 & & & & 0 & & \\
 & 0 \arrow[dr] & & H_n(X^{n+1}) \arrow[ur] \arrow[r, phantom, "\overset{(ii)}{\cong}"] & H_n(X) & & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & & {} \\
{} \arrow[r] & H_{n+1}(X^{n+1},X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "\delta_{n+1}"] \arrow[d, equal] & & H_n(X^n,X^{n-1}) \arrow[rr, "\delta_n"] \arrow[d, equal] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{n-1},X^{n-2}) \arrow[r] \arrow[d, equal] \arrow[ur] & {} \\
 & C_{n+1}^{\mathrm{CW}}(X) & & C_n^{\mathrm{CW}}(X) & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] \arrow[dr] & C_{n-1}^{\mathrm{CW}}(X) & \\
 & & & 0 \arrow[ur] & & {} &
\end{tikzcd}$$

☐

Example 2.2.7.2. Complex projective space $\mathbb{CP}^n = \{$1-dimensional subspaces of $\mathbb{C}^{n+1}\} = S^{2n+1}/\{z \sim \lambda z \mid \lambda \in S^1, z \in S^{2n+1}\} = (\mathbb{C}^{n+1} \smallsetminus 0)/\{z \sim \lambda z \mid z \in \mathbb{C}^{n+1} \smallsetminus 0, \lambda \in \mathbb{C}^*\}$. $\mathbb{CP}^n$ is a closed manifold of dimension $2n$. $\mathbb{CP}^n$ is a CW complex with $\mathbb{CP}^n = e^0 \cup e^2 \cup \cdots \cup e^{2n-2} \cup e^{2n}$. (This is shown on the homework).

$$\begin{array}{ccccccccccccccccccc}
C_\bullet^{\mathrm{CW}}(\mathbb{CP}^n): & \mathbb{Z} & \leftarrow & 0 & \leftarrow & \mathbb{Z} & \leftarrow & 0 & \leftarrow & \mathbb{Z} & \leftarrow & \cdots & \longleftarrow & \mathbb{Z} & \longleftarrow & 0 & \longleftarrow & \mathbb{Z} \\
\text{level} & 0 & & 1 & & 2 & & 3 & & 4 & & \cdots & & 2n-2 & & 2n-1 & & 2n
\end{array}$$

The differential $\delta \equiv 0$ (due to sparseness). Hence

$$H_k(\mathbb{CP}^n) \cong H_n^{\mathrm{CW}}(\mathbb{CP}^n) = C_k^{\mathrm{CW}}(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & \text{if } 0 \leq k \leq 2n \text{ and } k \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.2.7.3. Real projective space $\mathbb{RP}^n$ is a closed manifold of dimension n. Then $\mathbb{RP}^n$ is a CW complex, with $\mathbb{RP}^n = e^0 \cup e^1 \cup \cdots \cup e^n$. Then

$$\begin{array}{ccccccccccccccccc}
C_\bullet^{\mathrm{CW}}: & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \leftarrow & \mathbb{Z} & \leftarrow & \mathbb{Z} & \leftarrow & \cdots & \longleftarrow & \mathbb{Z} & \longleftarrow & \mathbb{Z} \\
\text{level} & 0 & & 1 & & 2 & & 3 & & 4 & & \cdots & & n-1 & & n
\end{array}$$

Lemma 2.2.7.4 (Cellular boundary formula)**.** *Let X be a CW complex. Then* $\delta_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$, *where* $d_{\alpha\beta} = \deg\left(S^{n-1} \xrightarrow{\varphi^n_\alpha} X^{n-1} \xrightarrow{q} X^{n-1}/X^{n-2} \xrightarrow{q_\beta} D^{n-1}_\beta/\partial D^{n-1}_\beta \cong S^{n-1}\right)$

Proof. This follows from diagram chasing in the following commutative diagram.

$$\begin{tikzcd}
\text{generator } [D^n_\alpha] \arrow[rr, mapsto] \arrow[ddddd, bend right, mapsto] & & \deg(q_\beta \circ q \circ \varphi^n_\alpha) e^{n-1}_\beta \arrow[ddrr, equal] & & \\
\Cap & & \Cap & & \\
H_n(D^n_\alpha, \partial D^n_\alpha) \arrow[r, "\partial", "\cong"'] \arrow[d, "(\Phi_\alpha)_*"] & \widetilde{H}_{n-1}(\partial D^n_\alpha) \arrow[r] \arrow[d, "(\varphi_\alpha)_*"] & \widetilde{H}_{n-1}(S^{n-1}_\beta) & \in & d_{\alpha\beta} e^{n-1}_\beta \\
H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[ddr, "\delta_n"'] & \widetilde{H}_{n-1}(X^{n-1}) \arrow[r, "q_*"] \arrow[dd, "j_{n-1}"] & \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}) \arrow[u, "(q_\beta)_*"'] \arrow[dd, "\cong"] & & \\
\in & & & & \\
e^n_\alpha \arrow[ddr, mapsto] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "\cong"] & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2}) & & \\
& \Cup & & & \\
& \sum_\beta d_{\alpha\beta} e^n_\beta \arrow[uuuuurrr, bend right, mapsto] & & &
\end{tikzcd}$$

$\square$

Example 2.2.7.5 (Example 2.2.7.3, continued)**.** Recall $\mathbb{RP}^n \cong \mathbb{RP}^{n-1} \cup_\varphi D^n$, where the attaching map $\varphi\colon S^{n-1} \to \mathbb{RP}^{n-1}$ is the projection. By the cellular boundary formula,

$$\delta_n(e^n) = \deg\left(\begin{tikzcd} S^{n-1} \arrow[r, "\varphi"] \arrow[rr, bend right, "f"'] & \mathbb{RP}^{n-1} \arrow[r, two heads, "q"] & \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2} \cong S^{n-1} \end{tikzcd}\right) e^n$$

Then for $y = [0, \dots, 1]$, we have $f^{-1}(y) = \{x = (0, 0, \dots, 1), -x\}$. Applying the degree theorem, $\deg(f) = \deg(f, x) + \deg(f, -x)$. Now, $\deg(f, x) \in \{\pm 1\}$ since f is a local homeomorphism, so $\deg(f, -x) = \deg(f \circ A, -x) = \deg(f, A(-x)) \cdot \deg(A, -x) = \deg(f, x) \cdot (-1)^n$, and hence

$$\deg(f) = \deg(f, x)(1 + (-1)^n) = \begin{cases} \pm 2 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Then by the diagram

$$\begin{array}{lccccccccccccccc}
C^{\mathrm{CW}}_\bullet: & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \leftarrow & \mathbb{Z} & \leftarrow & \mathbb{Z} & \leftarrow & \cdots & \longleftarrow & \mathbb{Z} & \longleftarrow & \mathbb{Z} \\
\text{degree} & 0 & & 1 & & 2 & & 3 & & 4 & & \cdots & & n-1 & & n
\end{array}$$

we conclude

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & \text{if } k = n \text{ is odd or } k = 0, \\ \mathbb{Z}/(2) & \text{if } k \text{ is odd and } 0 < k < n, \\ 0 & \text{otherwise.} \end{cases}$$

2.2.8. **Moore spaces: Spaces with prescribed homology.** Here we follow the beginning of [Hat02, Example 2.40]. For an abelian group π and $n \geq 1$, we construct a CW complex X such that $H_n(X) = \pi$ and $\widetilde{H}_i(X) = 0$ for $i \neq n$, called a *Moore space* or an *$M(\pi, n)$-space*. These are constructed (and usually assumed) to be simply connected when $n \geq 2$.

Note that a Moore space $M(S^n, n)$ is often called a *homology n-sphere*.

For $\pi = \mathbb{Z}/(m)$, take $X = S^n$ with an $(n+1)$-cell e^{n+1} attached by a degree m map $f \colon S^n \to S^n$. Recall this means m is the unique integer satisfying $f_*[i_n] = m[i_n] \in H_n(S^n) \cong \mathbb{Z}$. Such a map can be constructed by [Hat02, Exercise 2.31, p. 136]. More generally, any finitely generated $\mathbb{Z}$-module M can be constructed by taking wedge sums of this type for finite cyclic summands of π, together with copies of S^n for infinite cyclic summands of π. One can check the resulting space is an $M(\pi, n)$-space. Moore spaces can be constructed when π is not finitely generated too; see [Hat02, Example 2.40, pp. 143–4].

Corollary 2.2.8.1. *By taking wedge sums of Moore spaces, we can construct a space X with any prescribed homology $\{H_n X\}_{n\geq 1}$.*

2.3. Homology with coefficients

2.3.1. **Homology with coefficients.** Recall $C_q(X) = \mathbb{Z}[S_q(X)]$, where $S_q(X)$ is the set of all singular q-simplices $\sigma \colon \Delta^q \to X$. For $M \in \mathbb{Z}Mod$, define *q-chains of X with coefficients in M* by

$$C_q(X;M) \coloneqq M[S_q(X)] = \left\{ \sum_{\text{finite}} m_i \otimes \sigma_i \;\middle|\; m_i \in M, \sigma_i \in S_q(X) \right\} \cong C_q(X) \otimes M = \begin{array}{c}\text{free } \mathbb{Z}\text{-module}\\ \text{generated by } c\otimes m,\\ \text{for } c\in C_q(X), m\in M.\end{array}$$

Here, $m \cdot c = \sum (k_i m)\sigma_i$. The differential $d \colon C_q(X;M) \to C_{q-1}(X;M)$ is defined by the corresponding map $d \otimes 1_M \colon C_q(X) \otimes M \to C_{q-1}(X) \otimes M$. This gives a chain complex $C_\bullet(X;M)$, called the *singular chain complex with coefficients in M*. Now define *homology of X with coefficients in M* by $H_q(X;M) \coloneqq H_q(C_\bullet(X;M))$. We write $H_q(X, A; M)$ for the homology of the pair (X, A) with coefficients in M.

The following proposition follows from the fact that $C_q \otimes (M \oplus N) \cong C_q \otimes M \oplus C_q \otimes N$.

Proposition 2.3.1.1. *For $M, N \in \mathbb{Z}Mod$, $H_q(X; M \oplus N) \cong H_q(X;M) \oplus H_q(X;N)$.*

Our main interests lie in the cases when M is a commutative ring such as $\mathbb{Z}$, $\mathbb{Z}/(k)$ for $k \in \mathbb{Z}$, or fields such as $\mathbb{Z}/(p)$ for prime $p \in \mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$. Usually $H_\bullet(X;k)$ for a field k is much simpler than $H_\bullet(X) = H_\bullet(X;\mathbb{Z})$. The hope here is that we can somehow construct $H_\bullet(X) = H_\bullet(X;\mathbb{Z})$ from different $H_\bullet(X;M)$ that are easier compute.

Remark 2.3.1.2. Here we show how to calculate $H_q(X;M)$ from $C_\bullet^{\mathrm{CW}}(X)$ for a CW complex X. To do this, we make three observations.

(i) First note $C_\bullet^{\mathrm{CW}}(X)$ and $C_\bullet(X)$ are chain homotopic.

(ii) The assignment $C_\bullet \mapsto C_\bullet \otimes M$ gives a functor $\mathrm{Ch}(\mathbb{Z}Mod) \to \mathrm{Ch}(\mathbb{Z}Mod)$.

(iii) $\{T_q\}$ is a chain homotopy from f to g as chain maps $C_\bullet \to D_\bullet$. Indeed, $\partial T + T\partial = g - f$, so $T_q \otimes 1_M$ is a chain homotopy from $f \otimes 1_M$ to $g \otimes 1_M$. Thus $C_\bullet(X) \otimes M$ and $C_\bullet^{\mathrm{CW}}(X) \otimes M$ are chain homotopic.

We conclude that $H_\bullet(C_\bullet^{\mathrm{CW}}(X) \otimes M) \cong H_\bullet(X; M)$.

Example 2.3.1.3. $X = \mathbb{R}P^n = e^0 \cup e^1 \cup e^2 \cup \cdots \cup e^n$. Hence $H_q(\mathbb{RP}^n; \mathbb{Z}/(2)) = \delta_{0 \le q \le n}\mathbb{Z}/(2)$. More generally, for a field k,

$$H_q(\mathbb{RP}^n; k) = \begin{cases} k & q = 0, n \text{ for } n \text{ odd}, \\ 0 & q = 1, 2, \dots, n \text{ for } n \text{ even}, \\ 0 & \text{otherwise.} \end{cases}$$

2.3.2. **Universal coefficient theorem.** Still motivated by our search for a relation between $H_\bullet(X; M)$ and $H_\bullet(X)$, it is a good idea to explore how the algebraic counterpart $H_\bullet(C_\bullet \otimes M)$ is related to $H_\bullet(C_\bullet)$ for a general chain complex $C_\bullet$. There is an obvious homomorphism $\Phi\colon H_q(C_\bullet \otimes M \to H_q(C_\bullet \otimes M)$ given by $[c] \otimes m \mapsto [c \otimes m]$. We thus want to know if and when Φ an isomorphism. Recall that if $C_\bullet$ is a chain complex, then $C_\bullet$ is exact at C_q if and only if $H_q(C_\bullet) = 0$. Then our question becomes the following: if $0 \to C_2 \to C_1 \to C_0 \to 0$ is exact, then is $0 \to C_2 \otimes M \to C_1 \otimes M \to C_0 \otimes M \to 0$ also exact? If true, then $H_\bullet(C_\bullet \otimes M)$ is also zero.

Unfortunately, this is false in general. For example, $C_\bullet = 0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2 \to 0$ tensors by M to get $C_\bullet \otimes \mathbb{F}_2 = \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$, but the zero map is not injective. Luckily, this is the only issue in the sense that turns out that tensoring by M in general preserves exactness of a SES except injectivity of the left map, so we say $\otimes$ is *right exact*.

An exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is *split* if there exists a map $s\colon C \to B$ such that $g \circ s = \mathrm{id}_C$. *Not every short exact sequence splits!* Indeed, although the sequence $0 \to \mathbb{Z}/(3) \to \mathbb{Z}/(6) \to \mathbb{Z}/(2) \to 0$ splits (by the Chinese remainder theorem), the exact sequence $0 \to \mathbb{Z}/(2) \to \mathbb{Z}/(4) \to \mathbb{Z}/(2) \to 0$ is not split since $\mathbb{Z}/(4)$ is cyclic whereas the direct sum $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$ is not.

Proposition 2.3.2.1. *A short exact sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *splits if and only if there exists a map* $t\colon B \to A$ *such that* $t \circ f = \mathrm{id}_A$.

Proof. The reverse implication is easier to show, so we show the forward implication. If the map $s\colon C \to B$ is the map given by the split, then define $t\colon b \mapsto f^{-1}(b - sg(b))$. This map is well-defined because the $g \circ f \equiv 0 \implies g(f(f^{-1}(b - sg(b))) = 0$, so by exactness there is some element in the preimage of $fb - sg(b) \in B$ under f. But f is injective by exactness, so there is precisely one preimage, and hence t is well-defined. And indeed, t satisfies $(t \circ f)(a) = t(f(a)) = a$. □

Corollary 2.3.2.2. *If a short exact sequence $0 \to A \to B \to C \to 0$ splits, then $B \cong A \oplus C$.*

Proof. Indeed, in the notation of the proof of the previous proposition, we have an isomorphism $B \cong A \oplus C$ sending $b \in B$ to $(t(b), g(b)) \in A \oplus C$ and $(a, c) \in A \oplus C$ to $f(a) + s(c) \in B$. □

The following theorem is known as the *universal coefficient theorem (UCT).*

Theorem 2.3.2.3 (Universal coefficient theorem (algebraic version))**.** *For $C_\bullet \in \mathrm{Ch}(\mathbb{Z}Mod)$and $M \in \mathbb{Z}Mod$, there exists a natural exact sequence of the form*

$$0 \longrightarrow H_q(C_\bullet) \otimes M \xrightarrow{\Phi} H_q(C_\bullet \otimes M) \longrightarrow \overset{\mathbb{Z}}{\underset{1}{\mathrm{Tor}}}(H_{q-1}(C_\bullet), M) \longrightarrow 0.$$

Moreover, this sequence splits, but the splitting is not natural.

The following list summarizes the properties of $\mathrm{Tor}_1^{\mathbb{Z}}$ for use in computations; its formal definition will be given later.

- (i) $\mathrm{Tor}_0^{\mathbb{Z}}(M, N) = M \otimes N$ and $\mathrm{Tor}_1^{\mathbb{Z}}(M, N)$
- (ii) $\mathrm{Tor}_1^{\mathbb{Z}}(M \oplus M', N) \cong \mathrm{Tor}_1^{\mathbb{Z}}(M, N) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(M', N)$, and similarly in the other slot (similar to $(M \oplus M') \otimes N \cong (M \otimes N) \oplus (M' \otimes N)$).
- (iii) $\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}) = \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, M) = 0$ (similar to $M \otimes \mathbb{Z} \cong \mathbb{Z} \otimes M \cong M$).
- (iv) $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/k, \mathbb{Z}/\ell) \cong \mathbb{Z}/\mathrm{gcd}(k, \ell)$ (similar to $\mathbb{Z}/k \otimes \mathbb{Z}/\ell \cong \mathbb{Z}/\gcd(k, \ell)$).

Example 2.3.2.4 (Noncommutative rings)**.** It is sometimes useful to also work with noncommutative rings. For example, if $g \in \pi := \pi_1(X)$, then the elements of the group ring $R = \mathbb{Z}[\pi]$ are formal finite sums $\sum_i k_i g_i$ where $g_i \in \pi$ and $k_i \in \mathbb{Z}$. Since π is not abelian in general, the ring $\mathbb{Z}[\pi]$ need not be commutative. Where $p\colon \widetilde{X} \to X$ is the universal cover of X, g determines a deck transformation, which is a commutative diagram of the following form.

$$\begin{tikzcd}
\widetilde{X} \arrow[rr, "g"] \arrow[dr, "p"'] & & \widetilde{X} \arrow[dl, "p"] \\
& X &
\end{tikzcd}$$

It turns out that $C_q(\widetilde{X})$ is a left R-module. To show this, we use (a left-module analog of) the following straightforward observation: If R is a ring and M is an abelian group, then M is an R-module if there exists a group homomorphism $R \otimes M \to M$. In this setting, we can consider the map $\mathbb{Z}[\pi] \otimes C_q(\widetilde{X}) \longrightarrow C_q(\widetilde{X})$ given by $(\sum_i k_i g_i) \otimes c \longmapsto \sum_i k_i g_{i\sharp(c)}$, where $g_{i\sharp}$ is the chain map $C_q(\widetilde{X}) \to C_q(\widetilde{X})$ induced by the element $g_i \in \pi$.

2.3.3. **Free resolutions.** Let R be a (possibly noncommutative) ring and let $M \in ModR$. A *free resolution* $M_\bullet$ of M is an exact sequence $0 \leftarrow M \overset{\varepsilon^M}{\leftarrow} M_\bullet$ of free right R-modules of the form

$$0 \longleftarrow M \xleftarrow{\varepsilon^M} \underbrace{M_0 \xleftarrow{d_1} M_1 \xleftarrow{d_2} M_2 \longleftarrow \cdots}_{M_\bullet}.$$

We can think of a free resolution of M as a chain map ε^M from the chain complex $M_\bullet$ to the trivial chain complex $\cdots \leftarrow 0 \leftarrow M \leftarrow 0 \leftarrow \cdots$. In this perspective, we have the following commutative diagram.

$$\begin{tikzcd}
\cdots & 0 \arrow[l] & M \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & \cdots \arrow[l] \\
\cdots & 0 \arrow[l] \arrow[u, "0"] & M_0 \arrow[l] \arrow[u, "\varepsilon^M"] & M_1 \arrow[l, "d_1"'] \arrow[u, "0"] & M_2 \arrow[l, "d_2"'] \arrow[u, "0"] & \cdots \arrow[l]
\end{tikzcd}$$

It turns out that the chain map ε induces an isomorphism of the homologies of the trivial chain complex M and the chain complex $M_\bullet$.

Lemma 2.3.3.1. *Let $0 \leftarrow M \xleftarrow{\varepsilon} M_\bullet$ and $0 \leftarrow M' \xleftarrow{\varepsilon'} M'_\bullet$ be free resolutions and $f\colon M \to M'$ an R-module homomorphism.*

(i) *There is a chain map $f_\sharp$ making the following diagram commute.*

$$\begin{tikzcd}
M \arrow[d, "f"'] & M_\bullet \arrow[l, "\varepsilon"'] \arrow[d, "f_\sharp"] \\
M' & M'_\bullet \arrow[l, "\varepsilon'"]
\end{tikzcd}$$

(ii) *If $f_\sharp$ and $f'_\sharp$ are two chains maps making the diagram from part (i) commute, then $f_\sharp$ and $f'_\sharp$ are chain homotopic.*

Corollary 2.3.3.2. *If $0 \leftarrow M \xleftarrow{\varepsilon} M_\bullet$ and $0 \leftarrow M \xleftarrow{\varepsilon'} M'_\bullet$ are two free resolutions of M, then $\mathrm{id}_\sharp\colon M_\bullet \to M'_\bullet$ is a chain homotopy equivalence.*

Corollary 2.3.3.3. *If $N \in RMod$ then $\mathrm{id}_\sharp \otimes \mathrm{id}_N\colon M_\bullet \otimes_R N \to M'_\bullet \otimes_R N$ is a chain homotopy equivalence. Here $M_\bullet \otimes_R N$ is a chain complex of $\mathbb{Z}$-modules, and hence $H_q(M_\bullet \otimes_R N) \xrightarrow{\cong} H_q(M'_\bullet \otimes_R N)$ is an isomorphism, and this isomorphism is independent of the choice of chain map $\mathrm{id}_\sharp$ induced by the identity id.*

Proof of Corollary 2.3.3.3. By both parts of Lemma 2.3.3.1, we have the commutative diagram of chain complexes and chain maps

$$\begin{tikzcd}
 & M_\bullet \arrow[dl, "\varepsilon"'] \arrow[d, "f_\sharp"] \arrow[dr, bend left, dash] & & & & M_\bullet \otimes_R N \arrow[d, "f_\sharp \otimes 1"'] \arrow[dr, bend left, dash] & \\
M & M'_\bullet \arrow[l, "\varepsilon'"'] \arrow[d, "g_\sharp"] & \mathrm{id}_{M_\bullet \sharp} \arrow[l, squiggly, "T"'] \arrow[dl, bend left] & \Longrightarrow & & M'_\bullet \otimes_R N \arrow[d, "g_\sharp \otimes 1"'] & \mathrm{id}_{M_\bullet \otimes_R N \sharp} \arrow[l, squiggly, "T \otimes 1"'] \arrow[dl, bend left] \\
 & M_\bullet \arrow[ul, "\varepsilon"] & & & & M_\bullet \otimes_R N &
\end{tikzcd}$$

Here T is a chain homotopy from $\mathrm{id}_{M_{\bullet\sharp}}$ to $g_\sharp \circ f_\sharp$. Also the composition

$$H_q(M_\bullet \otimes_R N) \xrightarrow{(f_\sharp \otimes 1)_\bullet} H_q(M'_\bullet \otimes_R N) \xrightarrow{(g_\sharp \otimes 1)_\bullet} H_q(M_\bullet \otimes_R N)$$

is the identity. Similarly, $(f_\sharp \otimes 1)_\bullet \circ (g_\sharp \otimes 1)_\bullet = \mathrm{id}$. It follows that $(f_\sharp \otimes 1)_* \colon H_q(M_\bullet \otimes_R N) \to H_q(M'_\bullet \otimes_R N)$ is an isomorphism. □

2.3.4. **The Tor functor and example computations.** For a (possibly noncommutative) ring R, define the *Tor functor* by $\mathrm{Tor}_q^R(M,N) := H_q(M_\bullet \otimes_R N)$, where $0 \leftarrow M \overset{\varepsilon^M}{\leftarrow} M_\bullet$ is *any* free resolution of M.

(i) Suppose we have a free resolution $0 \leftarrow M \xleftarrow{\varepsilon^m} M_\bullet$. If $q = 0$ then we can just use the short exact sequence

$$0 \longleftarrow M \xleftarrow{\varepsilon^M} \underbrace{M \xleftarrow{d_1} M_1/\ker \varepsilon^M \longleftarrow 0 \leftarrow 0 \leftarrow \cdots}_{M_\bullet}$$

as the free resolution of M. We thus want to compute

$$H_0\left(0 \longleftarrow M_0 \otimes_R N \xleftarrow{d_1 \otimes 1_N} M_1/\ker d_1 \longleftarrow 0\right)$$

Applying the right exact functor $(-) \otimes_R N$ gives an exact sequence

$$0 \longleftarrow M \otimes_R N \xleftarrow{\varepsilon^M \otimes 1_N} M_0 \otimes_R N \xleftarrow{d_1 \otimes 1_N} (M_1/\ker d_1) \otimes_R N$$

By exactness, $\varepsilon^M \otimes 1_N$ is surjective. Hence, by the first isomorphism theorem, $M \otimes_R N \cong (M_0 \otimes_R N)/\ker(\varepsilon^M \otimes 1_N)$. By exactness we have $\ker(\varepsilon^M \otimes 1_N) = \mathrm{im}(d_1 \otimes 1_N) = B_0(M_\bullet \otimes_R N) =: B_0$. Moreover, $Z_0 := Z_0(M_\bullet \otimes_R N) = \ker(0 \leftarrow M_0 \otimes_R N) = M_0 \otimes_R N)$. Hence we can write

$$M \otimes_R N \cong (M_0 \otimes_R N)/\mathrm{im}(d_1 \otimes 1_N) \cong Z_0/B_0 =: H_0(M_\bullet \otimes_R N) =: \underset{0}{\overset{\mathbb{Z}}{\mathrm{Tor}}}(M,N).$$

(i) If the ring is a field k then any k-module is a vector space, which is free. In other words, every module over a field is free, so any exact sequence of the form $0 \leftarrow M \leftarrow M_\bullet$ is a free resolution of M. Hence we can take the resolution $0 \leftarrow M \overset{\mathrm{id}}{\leftarrow} M \leftarrow 0$ and apply $(-) \otimes_k N$ to obtain the chain complex $0 \leftarrow M \otimes_R \leftarrow 0$ with trivial differential and vanishing groups at all degrees except zero. Therefore, for all k-vector spaces V and W, $\mathrm{Tor}_q^k(V,W) = \delta_{q=0} V \otimes_k W$.

(ii) Consider the case the ring R is the integers $\mathbb{Z}$. If M is any $\mathbb{Z}$-module, then there is always a short free resolution given by

$$\begin{tikzcd}
0 & M \arrow[l] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \mathbb{Z}[S] \arrow[l, two heads, "\varepsilon^M"'] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \ker \varepsilon^M \arrow[l, hook'] & 0 \arrow[l] & \arrow[l] \\
 & \sum_i k_i s_i & \sum_i k_i s_i \arrow[l, mapsto] & & &
\end{tikzcd}$$

where S is the set of generators of M. This is a free resolution, since $\ker\varepsilon^M$ is a submodule of a free $\mathbb{Z}$-modules and any submodule of a free $\mathbb{Z}$-module is free.

(iii) More generally, if R is a principal ideal domain (PID), then any submodule of a free R-module is free. Hence there is a short free resolution of the form

$$\begin{tikzcd}
0 & M \arrow[l] & R[S] \arrow[l, two heads, "\varepsilon^M"'] & \ker\varepsilon^M \arrow[l, hook] & 0 \arrow[l] & \arrow[l] \\
 & \sum_i k_i s_i \arrow[u, phantom, sloped, "\in"] & \sum_i k_i s_i \arrow[l, mapsto] \arrow[u, phantom, sloped, "\in"] & & &
\end{tikzcd}$$

where S is the set of generators of M. Tensoring with N,

$$M_\bullet \otimes_R N\colon \qquad 0 \longleftarrow R[S]\otimes_R N \xleftarrow{i\otimes 1_N} \ker\varepsilon^M \otimes_R N \longleftarrow 0,$$

where $i\colon \ker\varepsilon^M \hookrightarrow R[S]$ is the inclusion map. Then take homology to conclude

$$\underset{q}{\overset{R}{\mathrm{Tor}}}(M,N) = \begin{cases} M\otimes_R N & \text{if } q=0, \\ \ker(i\otimes 1_N) & \text{if } q=1, \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.3.4.1. Let $R=\mathbb{Z}$, $M=\mathbb{Z}/(2)$. There is a free resolution $\mathbb{Z}\xleftarrow{\varepsilon^M} \underbrace{\mathbb{Z} \xleftarrow{\times 2} \mathbb{Z}}_{M_\bullet}$. Then $H_q(\mathbb{Z}/(2) \cong \mathbb{Z}\otimes\mathbb{Z}/(2) \xleftarrow{2\otimes 1\equiv 0} \mathbb{Z}/(2)) = \delta_{q=0,1}\mathbb{Z}/(2)$.

Theorem 2.3.4.2 (Universal coefficient theorem (UCT))**.** *If R is a (commutative) PID, $M\in RMod$, and $C_\bullet$ is a chain complex of free R-modules, then there is a natural SES of the form*

$$0 \longrightarrow H_q(C_\bullet)\otimes_R M \longrightarrow H_q(C_\bullet\otimes_R M) \longrightarrow \underset{1}{\overset{R}{\mathrm{Tor}}}(H_{q-1}(C_\bullet), M) \longrightarrow 0$$

that splits, but not naturally.

2.3.5. **Künneth theorem for homology.** A *Künneth theorem* is a statement relating the homology of two objects to the homology of their product, so the Künneth theorem below should compute $H_\bullet(X\times Y)$ in terms of $H_\bullet(X)$ and $H_\bullet(Y)$. We first consider the case of CW complexes. If X and Y are CW complexes, one may ask if we can compute $C_\bullet^{\mathrm{CW}}(X\times Y)$ in terms of $C_\bullet^{\mathrm{CW}}(X)\otimes C_\bullet^{\mathrm{CW}}(Y)$. Recall that $C_m^{\mathrm{CW}}(X) = \mathbb{Z}[S_m^{\mathrm{CW}}(X)]$, where $S_m^{\mathrm{CW}}(X)$ denotes the set of m-cells e_α^m in X. Thus

$$\begin{aligned} C_q^{\mathrm{CW}}(X\times Y) &= \mathbb{Z}[S_q^{\mathrm{CW}}(X\times Y)] = \mathbb{Z}\left[\coprod_{m+n=q} S_m^{\mathrm{CW}}(X)\times S_n^{\mathrm{CW}}(Y)\right] \\ &= \bigoplus_{m+n=q} \mathbb{Z}[S_m^{\mathrm{CW}}(X)]\otimes\mathbb{Z}[S_n^{\mathrm{CW}}(Y)] = \bigoplus_{m+n=q} S_m^{\mathrm{CW}}(X)\otimes S_n^{\mathrm{CW}}(Y), \end{aligned}$$

where the second equality follows from the straightforward observation that $\mathbb{Z}[S \amalg T] = \mathbb{Z}[S] \oplus \mathbb{Z}[T]$, the third equality follows the fact that the map $\mathbb{Z}[S] \otimes \mathbb{Z}[T] \to \mathbb{Z}[S \times T]$ given by $(\sum_\alpha k_\alpha s_\alpha) \otimes (\sum_\beta \ell_\beta t_\beta) \mapsto \sum_{\alpha,\beta} k_\alpha \ell_\beta (s_\alpha, t_\beta)$ is an isomorphism, and the fourth equality follows from the definition of $C_m^{\mathrm{CW}}(X)$ and $C_n^{\mathrm{CW}}(Y)$. This motivates the following definition.

If $C_\bullet$ and $D_\bullet$ are chain complexes, we define the *tensor product chain complex* of $C_\bullet$ of $D_\bullet$ by $(C_\bullet \otimes D_\bullet)_q := \bigoplus_{m+n=q} C_m \otimes D_n$ for each $q \in \mathbb{Z}$, with boundary maps $\partial_q \colon (C_\bullet \otimes D_\bullet)_q \to (C_\bullet \otimes D_\bullet)_{q-1}$ defined first on the generators $c \otimes d$ of the summands $C_m \otimes D_m$ of $(C_\bullet \otimes D_\bullet)_q$ by $\partial_q(c \otimes d) := \partial^C c \otimes d + (-1)^m c \otimes \partial^D d$, and extending ∂_q linearly to all of $(C_\bullet \otimes D_\bullet)_q$.

Theorem 2.3.5.1 (Künneth theorem (algebraic version)). *For a PID R and chain complexes $C_\bullet, D_\bullet \in \mathrm{Ch}(RMod)$ with $C_\bullet$ free over R, there is a SES of the form*

$$0 \longrightarrow (H_\bullet C_\bullet \otimes_R H_\bullet D_\bullet)_q \xrightarrow{\times} H_q(C_\bullet \otimes_R D_\bullet) \longrightarrow (\underset{q}{\overset{R}{\mathrm{Tor}}}(H_\bullet C_\bullet, H_\bullet D_\bullet)_{q-1} \longrightarrow 0,$$

where $\times$ is the cross product *given by $[c] \otimes [d] \mapsto [c \otimes d]$, $(\mathrm{Tor}_q^R(H_\bullet C_\bullet, H_\bullet D_\bullet))_{q-1}$ denotes $\bigoplus_{m+n=q-1} \mathrm{Tor}_1^R(H_m C_\bullet, H_n D_\bullet)$, and $(H_\bullet C_\bullet \otimes_R H_\bullet D_\bullet)_q$ is defined above.*

Note that if $D_q = 0$ for $q \neq 0$, the Künneth theorem is just the UCT.

The following corollary gives a nice way to formulate that $H_\bullet(C_\bullet \otimes_k D_\bullet) = H_\bullet C_\bullet \otimes_k H_\bullet D_\bullet$, where the left-hand side has trivial boundary maps, e.g., in characteristic 2.

Corollary 2.3.5.2. *Under the hypothesis of the Künneth theorem, if k is a field, then the cross product $\times \colon H_q(C_\bullet \otimes_k D_\bullet) \to \bigoplus_{m+n=q} H_m(C_\bullet) \otimes_k H_n(D_\bullet)$ given by $[c] \otimes [d] \mapsto [c \otimes d]$ is an isomorphism.*

2.3.6. **Cellular Künneth theorem.** If X, Y are CW complexes, then the map $C_\bullet^{\mathrm{CW}}(X) \otimes C_\bullet^{\mathrm{CW}}(Y) \xrightarrow{\times} C_\bullet^{\mathrm{CW}}(X \times Y)$ given by $e_\alpha^m \otimes e_\beta^n \mapsto e_\alpha^m \times e_\beta^n$ is an isomorphism of chain complexes.

Corollary 2.3.6.1 (Cellular Küneth Theorem). *If X, Y are CW complexes, then there is short exact sequence that splits, but not naturally, of the form*

$$0 \to (H_\bullet^{\mathrm{CW}}(X; R) \otimes_R H_\bullet^{\mathrm{CW}}(Y; R))_q \xrightarrow{\times} H_q^{\mathrm{CW}}(X \times Y; R) \to \underset{1}{\overset{R}{\mathrm{Tor}}}(H_\bullet^{\mathrm{CW}}(X), H_\bullet^{\mathrm{CW}}(Y))_{q-1} \to 0.$$

If X, Y are topological spaces, then there is always a natural chain map $C_\bullet(X) \otimes C_\bullet(Y) \xrightarrow{\times} C_\bullet(X \times Y)$, where $\times$ is defined as a sort of alternating sum defined in a similar way to that of the prism operator (from the proof of homotopy invariance of homology), that is a *chain homotopy equivalence*. In particular, $H_\bullet(C_\bullet(X \times Y)) \cong H_\bullet(X \times Y)$. Omitting the straightforward but tedious details, we obtain the following corollary.

Corollary 2.3.6.2 (Singular homology of product spaces). *If X, Y are topological spaces, then there exists a short exact sequence that splits, but not naturally, of the form*

$$0 \to (H_\bullet(X; R) \otimes_R H_\bullet(Y; R))_q \xrightarrow{\times} H_q(X \times Y; R) \to \underset{1}{\overset{R}{\mathrm{Tor}}}(H_\bullet(X), H_\bullet(Y))_{q-1} \to 0.$$

2.3.7. **Poincaré series.** Poincaré series provide an easy way to record information of the homology with coefficients in a field for many spaces. If X is of *finite type*, i.e., if $H_q(X)$ is finitely generated for each q, then we can define the *Poincaré series* of X with coefficients in the field $\mathbb{K}$ by $P(X;\mathbb{K}) \coloneqq \sum_{q=0}^{\infty} \dim_{\mathbb{K}}(H_q(X;\mathbb{K}))t^q$. If $\mathbb{K}$ is a field then $H_q(X;\mathbb{K})$ is a vector space, in which case it is determined up to isomorphism by its vector space dimension.

Lemma 2.3.7.1. *$P(X \amalg Y;\mathbb{K}) = P(X;\mathbb{K}) + P(Y;\mathbb{K})$. $P(X \times Y;\mathbb{K}) = P(X;\mathbb{K}) \cdot P(Y;\mathbb{K})$.*

Some examples include $P((\mathbb{RP}^n)^{\times k};\mathbb{K}) = ((1-t^{n+1})/(1-t))^k = (1+t+t^2+\cdots+t^n)^k$, $P(S^n;\mathbb{K}) = t+t^n$, $P(S^n \vee S^m;\mathbb{K}) = 1+t^n+t^m$, and so on, so we can always find a topological space whose Poincaré series is any given power series.

2.4. (SINGULAR) COHOMOLOGY

2.4.1. **(Singular) cohomology.** Let $M \in \mathbb{Z}Mod$ and fix a chain complex $C_\bullet$, which we will index with *descending* labels so that $C_\bullet$ is $\cdots \longrightarrow C_{q+1} \xrightarrow{\partial} C_q \xrightarrow{\partial} C_{q-1} \longrightarrow \cdots$. Define the *dual* chain complex $C^\bullet$ by $C_\bullet^\vee \coloneqq \operatorname{Hom}_{\mathbb{Z}}(C_\bullet, M)$, so that $C^\bullet$ takes the following form.

$$C^\bullet: \qquad \cdots \longleftarrow \underset{\mathbb{Z}}{\operatorname{Hom}}(C_{q+1}, M) \xleftarrow{\delta=\partial^\vee} \underset{\mathbb{Z}}{\operatorname{Hom}}(C_q, M) \xleftarrow{\delta} \underset{\mathbb{Z}}{\operatorname{Hom}}(C_{q-1}, M) \longleftarrow \cdots.$$

In particular, $\delta \coloneqq \partial^\vee$ is defined by sending each $\varphi \in C^q$ to $\delta\varphi \coloneqq \delta^*\varphi = \varphi \circ \partial \in C^{q+1}$. Then $(C^\bullet, \delta)$ satisfies $\delta^2 = 0$, which is to say is a *cochain complex* (of $\mathbb{Z}$-modules), though of course formally chain complexes and cochain complexes are equivalent notions; this is only for psychological purposes.

Examples of cochain complexes include the following. (i) If $X \in \mathsf{Top}$ and $M \in \mathbb{Z}Mod$, then the *singular cochain complex* is given by $C^\bullet(X;M) \coloneqq \operatorname{Hom}(C_\bullet(X), M)$. (ii) If X is a smooth manifold, then the *de Rham complex*, denoted $\Omega^\bullet(X)$, is given by

$$\cdots \xrightarrow{\mathrm{d}} \Omega^{q-1}(X) \xrightarrow{\mathrm{d}} \Omega^q(X) \xrightarrow{\mathrm{d}} \Omega^{q+1}(X) \xrightarrow{\mathrm{d}} \cdots,$$

where d is the differential on X and $\Omega^q(X)$ denotes the differential q-forms on X.

Define the qth *cohomology group* of a cochain complex $C^\bullet$ by

$$H^q(C^\bullet) \coloneqq \ker(C^q \xrightarrow{\delta} C^{q+1}) / \operatorname{im}(C^{q-1} \xrightarrow{\delta} C^q).$$

The numerator consists of *q-cocycles*, and the denominator consists of *q-coboundaries*, denoted by $Z^q(C^\bullet)$ and $B^q(C^\bullet)$ respectively. The cohomology $H^q(\Omega^\bullet)$ is called *de Rham cohomology*. The cohomology $H^q(X;M) \coloneqq H^q(C^\bullet(X;M))$ is called *singular cohomology*. We also define cohomology of pairs analogously as we did in the homology case, i.e., $H^\bullet(X,A;M) \coloneqq \operatorname{Hom}(C_\bullet(X,A;M))$. Other other constructions go through without issue by dualizing everything we did for homology. For example, the following *Eilenberg–Steenrod axioms* hold, which we recount for completeness.

(1) Functoriality of $H^q(-;M)$: Given an $M \in \mathbb{Z}Mod$, any map $f\colon X \to Y$ induces a map $f^*\colon H^q(X;M) \to H^q(Y;M)$ given by $(f^*[\varphi])(\sigma) \coloneqq [\varphi(f(\sigma))]$. In this case, note that $H^q(-;M)$ is a *contravariant* functor.
(2) Homotopy invariance: If $f, g\colon X \to Y$ are homotopic, then the induced maps $f^*, g^*\colon H^q(Y;M) \to H^q(X;M)$ coincide.
(3) Additivity: $H^q(\coprod_\alpha X_\alpha; M) \cong \prod_\alpha H^q(X_q;M)$.
Warning. $\mathrm{Hom}(\bigoplus_\alpha N_\alpha; M) \cong \prod_\alpha \mathrm{Hom}(N_\alpha, M))$.
(4) Mayer-Vietoris LES $X = U \cup V$, $U, V \subseteq X$ open,

$$\leftarrow H^q(U \cap V) \leftarrow H^q(U) \oplus H^q(V) \leftarrow H^q(X) \xleftarrow{\delta} H^{q-1}(U \cap V) \leftarrow$$

(5) A triple (X, B, A) $(A \subseteq B \subseteq X)$ induces a long exact sequence of pairs of the form

$$\leftarrow H^q(B,A) \leftarrow H^q(X,A) \leftarrow H^q(X,B) \xleftarrow{\delta} H^{q-1}(B,A)$$

(6) Excision: If $Z \subseteq A \subseteq X$ and $\overline{Z} \subseteq A^\circ$, then there exists an isomorphism

$$H^q(X,A) \xrightarrow{\cong} H^q(X \smallsetminus Z, A \smallsetminus Z).$$

(7) If X is a CW complex, then the cellular chain complex $C_\bullet^{\mathrm{CW}}(X)$ is homotopy equivalent to the singular chain complex $C_\bullet(X)$. Hence $\mathrm{Hom}(C_\bullet^{\mathrm{CW}}(X), M)$ is homotopy equivalent to $\mathrm{Hom}(C_\bullet(X), M)$, so we also have $H^q_{\mathrm{CW}}(X;M) \cong H^q(X;M)$.

Example 2.4.1.1. Consider $\mathbb{RP}^n$. We have

$$\begin{tikzcd}
\text{degree} & 0 & 1 & 2 & 3 & 4 & \\
C_\bullet^{\mathrm{CW}}(\mathbb{RP}^n) & \mathbb{Z} & \mathbb{Z} \arrow[l] & \mathbb{Z} \arrow[l, "2"'] & \mathbb{Z} \arrow[l, "0"'] & \mathbb{Z} \arrow[l, "2"'] & {} \arrow[l, "0"'] \\
H_\bullet(\mathbb{RP}^n) & \mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \\
C^\bullet_{\mathrm{CW}}(\mathbb{RP}^n) & \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \arrow[r, "\delta"] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \arrow[r, "\delta"] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \arrow[r, "\delta"] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \arrow[r, "\delta"] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \arrow[r, "\delta"] \arrow[d, phantom, "\cong" rotate=90] & {} \\
 & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r, "0"] & {} \\
H^\bullet(\mathbb{RP}^n) & \mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} &
\end{tikzcd}$$

Here we used that the dual of multiplication by k is still multiplication by k. Indeed, this follows from the following commutative diagram.

$$\begin{tikzcd}
g \circ (\times k) \arrow[d, mapsto] & \in & \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \arrow[d, "\cong"'] & \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \arrow[l, "(\times k)^\vee"'] \arrow[d, "\cong"] & \in & g \arrow[d, mapsto] \arrow[lllll, mapsto, bend right] \\
\substack{(g\circ(\times k))(1) \\ =g(k)=k\cdot g(1)} & \in & \mathbb{Z} & \mathbb{Z} \arrow[l, "\times k"'] & \in & g(1) \arrow[lllll, mapsto, bend left]
\end{tikzcd}$$

The natural geometric question to ask is whether $H^\bullet(X;M)$ can be expressed in terms of $H_\bullet(X)$ and M. A more algebraic version of this is that, for a chain complex $C_\bullet$ of $\mathbb{Z}$-modules, whether it is true that $H^\bullet(\mathrm{Hom}_{\mathbb{Z}Mod}(C_\bullet, M))$ be expressed in terms of $H_\bullet(C_\bullet)$ and M. Even more generally, one could ask that, when R is any (possibly noncommutative) ring and $C_\bullet$ is a chain complex of (left or right) R-modules, whether $H^\bullet(\mathrm{Hom}_R(C_\bullet, M))$ be expressed in terms of the R-modules $H_\bullet(C_\bullet)$ and M.

Recall that if N, M are left R-modules, then although the set $\mathrm{Hom}_R(N, M)$ of R-module maps is a $\mathbb{Z}$-module, it is *not* an R-module in general, though it is an R-module if R is commutative. This is similar to the fact that when R is noncommutative, there is no guarantee that the tensor product of right and left R-modules is an R-module, but only that it is a $\mathbb{Z}$-module.

Let $C_\bullet$ be a chain complex of R-modules, and let M be an R-module. We always have the *evaluation map* given by

$$\begin{aligned} \mathrm{ev}\colon H^q(\underset{R}{\mathrm{Hom}}(C_\bullet, M)) &\longrightarrow \underset{R}{\mathrm{Hom}}(H_q(C_\bullet), M), \\ [\varphi\colon C_q \to M] &\longmapsto ([c] \mapsto \varphi(c)). \end{aligned}$$

Claim 2.4.1.2. *The evaluation map* ev *is well-defined.*

Proof. First we show ev is independent of c: If $c' = c + \partial b$, then $\varphi(c') = \varphi(c + \partial b) = \varphi(c) + \varphi(\partial b) = \varphi(c) + (\varphi \circ \partial)(b) = \varphi(c) + (\delta\varphi)(b)$. Passing to cohomology, the rightmost expression is $\varphi(c)$. Next we show ev is independent of the representative φ: If $\varphi' = \varphi + \delta\psi$, then $\varphi'(c) = \varphi(c) + (\delta\psi)(c) = \varphi(c) + (\psi \circ \partial)(c)$. Passing to cohomology, the rightmost expression is $\varphi(c)$ since c is a cycle. □

Theorem 2.4.1.3 (UCT for cohomology (algebraic version)). *For a PID R, $M \in RMod$, and a chain complex $C_\bullet$ of free R-modules, there is a natural SES of the form*

$$0 \longrightarrow \underset{R}{\overset{1}{\mathrm{Ext}}}(H_{q-1}(C_\bullet), M) \longrightarrow H^q(\underset{R}{\mathrm{Hom}}(C_\bullet, M)) \xrightarrow{\mathrm{ev}} \underset{R}{\mathrm{Hom}}(H_q(C_\bullet), M) \longrightarrow 0.$$

that splits, but not naturally.

Let $N \in ModR$. Then the functor $M \mapsto M \otimes_R N$ is a functor $ModR \to \mathbb{Z}Mod$ sending $f\colon A \to B$ to $f \otimes \mathrm{id}_M\colon A \otimes_R M \to B \otimes_R M$, which in general is not injective. Thus the tensor product is not exact, and in fact the tensor product is right exact. We then defined $\mathrm{Tor}_q^R(M, N) := H_q(M_\bullet \otimes_R N)$, where $M_\bullet$ is any free resolution of M.

The present situation is similar. Let $N \in RMod$. Then the functor $M \mapsto \mathrm{Hom}_R(M, N)$ is a *contravariant* functor $RMod \to \mathbb{Z}Mod$ sending $f\colon A \to B$ to $f^\vee\colon \mathrm{Hom}_R(B, N) \to \mathrm{Hom}_R(A, N)$, and this assignment is not surjective in general. And in fact the hom functor is left exact. This suggests the following definition for the "coTor" in the current setting, which for reasons mentioned below will be called Ext.

For any (possibly noncommutative) ring R, define $\operatorname{Ext}_R^q(M,N) := H^q(\operatorname{Hom}_R(M_\bullet, N))$ where $M_\bullet$ is *any* free resolution of M. The reason for the name is that it can be shown that $\operatorname{Ext}_R^1(M,N)$ classifies *extensions* of R-modules, that is, SESs of R-modules of the form $0 \to N \to \widehat{M} \to M \to 0$.

Lemma 2.4.1.4 (Properties of Ext). *For a (possibly noncommutative) ring R and $M, N \in RMod$, the following hold.*

(i) $\operatorname{Ext}_R^0(M,N) = \operatorname{Hom}_R(M,N)$.

(ii) *For a free R-module M, $\operatorname{Ext}_R^q(M,N) = 0$ for all $q \geq 1$.*

(iii) *For a field $\mathbb{K}$, $\operatorname{Ext}_{\mathbb{K}}^q(M,N) = 0$ for all $q \geq 1$.*

(iv) *If R is a PID, then $\operatorname{Ext}_R^q(M,N) = 0$ for all $q \geq 2$.*

(v) $\operatorname{Ext}_R^q(M \oplus M', N) = \operatorname{Ext}_R^q(M,N) \oplus \operatorname{Ext}_R^q(M',N)$*, and similarly for the second slot.*

(vi) $\operatorname{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/(k), \mathbb{Z}/(\ell)) \cong \mathbb{Z}/(d)$ *where $d = \gcd(k,\ell)$.*

(vii) $\operatorname{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/(k), \mathbb{Z}) \cong \mathbb{Z}/(k)$.

Applying the algebraic version of the UCT to $C_\bullet(X)$, we obtain the geometric counterpart of Theorem 2.4.1.3.

Theorem 2.4.1.5 (UCT for cohomology (geometric version)). *For a PID R and $M \in RMod$, there is a natural SES of the form*

$$0 \longrightarrow \underset{R}{\overset{1}{\operatorname{Ext}}}(H_{q-1}(X;R), M) \longrightarrow H^q(X;M) \xrightarrow{\text{ev}} \underset{R}{\operatorname{Hom}}(H_q(X;R), M) \longrightarrow 0$$

that splits, but not naturally.

In particular, if $\mathbb{K}$ is a field, then we have a $\mathbb{K}$-vector space isomorphism $\operatorname{ev}\colon H^q(X;\mathbb{K}) \xrightarrow{\cong} \operatorname{Hom}_{\mathbb{K}}(H_q(X;\mathbb{K}), \mathbb{K}) = H_q(X;\mathbb{K})^\vee$.

2.4.2. **The cup product on cohomology.** Fix a (possibly noncommutative) ring R. Define the *evaluation pairing* by the map $H^q(X;M) \times H_q(X;R) \to M$ given for $\gamma = [\varphi\colon C_q(X) \to M] \in H^q(X;M)$ and $h = [c] \in H_q(X;R)$ by $(\gamma, h) \mapsto \langle \gamma | h \rangle := \varphi(c) \in M$. We define the *cup product* first on cochains by

$$C^k(X;R) \times C^\ell(X;R) \xrightarrow{\smile} C^{k+\ell}(X;R) = \underset{R}{\operatorname{Hom}}(C_{k+\ell}(X), R),$$

$$(\varphi, \psi) \longmapsto \varphi \smile \psi,$$

where for each $\sigma\colon \Delta^{k+\ell} \to X$ in $C_{k+\ell}(X;R)$, we define $(\varphi \smile \psi)(\sigma)$ by the multiplication in R given by

$$(\varphi \smile \psi)(\sigma) := \varphi(\underbrace{\sigma \circ [v_0, \ldots, v_k]}_{\textit{“front } k\textit{-face of } \sigma\textit{”}}) \cdot \psi(\underbrace{\sigma \circ [v_k, \ldots, v_{k+\ell}]}_{\textit{“back } \ell\textit{-face of } \sigma\textit{”}}).$$

Lemma 2.4.2.1. $\delta(\varphi \smile \psi) = (\delta\varphi) \smile \psi + (-1)^{|\varphi|}\varphi \smile (\delta\psi)$*, where $|\varphi| = k$ if $\varphi \in C^k(X;R)$.*

As a consequence of the above lemma, we get a well-defined cup product in cohomology given by $H^k(X;R) \times H^\ell(X;R) \xrightarrow{\smile} H^{k+\ell}(X;R)$, $([\varphi],[\psi]) \mapsto [\varphi \smile \psi]$.

It follows at once from the definition of the cup product that it is associative, distributive, and unital, with unit give by $[\varepsilon] = 1 \in H^0(X;R)$, where we recall $\varepsilon\colon C_0(X) \to R$ is the *augmentation map* sending simplices to $1 \in R$. The following theorem shows that when R is commutative, the cup product is graded-commutative; see Hatcher's book for a proof.

Theorem 2.4.2.2. *If R is a commutative ring, $\alpha \in H^k(X;R)$, and $\beta \in H^\ell(X;R)$, then $\alpha \smile \beta = (-1)^{k\ell}\beta \smile \alpha$.*

Corollary 2.4.2.3. *If $\alpha \in H^k(X;R)$, k is odd, and 2 is invertible in R, then $\alpha \smile \alpha = 0$.*

Proposition 2.4.2.4. *The cup product is compatible with induced maps in the sense that if $f\colon X \to Y$, so that $f^*\colon H^\bullet(Y;R) \to H^\bullet(X;R)$, then $f^*(\alpha \smile \beta) = f^*\alpha \smile f^*\beta$.*

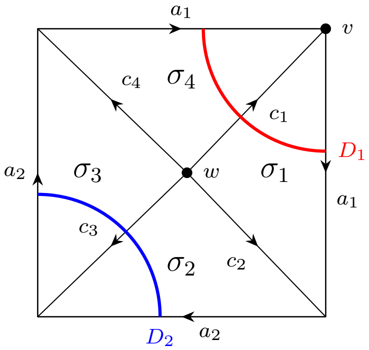
Example 2.4.2.5. Consider the closed connected surface $X_2 = \mathbb{RP}^2\#\mathbb{RP}^2$. We can write X_2 as a square with each edge oriented clockwise about the boundary of the square, with the bottom and left edges identified as the edge a_2 and the top and right edge identified as a_1, v the single vertex, and f the single face. Then, as we have already shown before,

$$H_q(X_2;\mathbb{F}_2) = \begin{cases} \mathbb{F}_2 v & \text{if } q = 0, \\ \mathbb{F}_2 a_1 \oplus \mathbb{F}_2 a_2 & \text{if } q = 1, \\ \mathbb{F}_2 f & \text{if } q = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Then by the geometric version of the universal coefficient theorem for cohomology, we have $H^q(X_2;\mathbb{F}_2) \cong H_q(X_2;\mathbb{F}_2)^\vee$, where the dual is as an $\mathbb{F}_2$-vector space. In the particular case $q = 1$, we have

$$H^1(X_2;\mathbb{F}_2) \cong H_1(X_2;\mathbb{F}_2)^\vee \cong (\mathbb{F}_2 a_1 \oplus \mathbb{F}_2 a_2)^\vee = \mathbb{F}_2\alpha_1 \oplus \mathbb{F}_2\alpha_2,$$

where α_1 and α_2 are the elements of $H^q(X_2;\mathbb{F}_2)$ dual to the elements a_1 and a_2 of $H_q(X_2;\mathbb{F}_2)$, respectively. We claim that $\alpha_1 \smile \alpha_2 = 1$, $\alpha_1 \smile \alpha_2 = 0$, and $\alpha_2 \smile \alpha_2 = 1$. But we have a problem—the cup product is defined on cohomology classes by evaluating them on the representative cocycles on *simplices*. So we need simplices to work with! To do this, divide the center face into four simplices as follows. (Note that we could probably do it with two, but dividing it up by adding a point w the center in this way lets us easily generalize this procedure to X_k for $k \geq 2$.)



Then we have $\varphi_i(a_j) = \delta_{ij}$ and $\varphi_i(\text{1-simplex } \sigma) = \#\{\text{intersection points of } \sigma \text{ with } D_i\}$. And φ_i is a cocycle since $(\delta\varphi_i)(\sigma_j) = \varphi_i(\partial\sigma_j) = 0$, and hence $[\varphi_i] = \alpha_i \in H^1(X_2; \mathbb{F}_2)$. Moreover, $H_2(X_2; \mathbb{F}_2) = \mathbb{F}_2[\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4]$. We now compute on the basis elements to obtain $\langle \alpha_1 \smile \alpha_1 | [\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4] \rangle = \sum_j (\varphi_1 \smile \varphi_1)(\sigma_j) = \varphi_1(\text{front 1-face of } \sigma_1)\varphi_1(\text{back 1-face of } \sigma_1) + \varphi_1(\text{front 1-face of } \sigma_4)\varphi_1(\text{back 1-face of } \sigma_4) = 1 \cdot 1 + 0 \cdot 1 = 1$. It then follows by duality that $\alpha_1 \smile \alpha_1 \neq 0$.

Example 2.4.2.6. Computing similarly to the previous example, we obtain

$$H^q(\mathbb{RP}^2; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & \text{if } q = 0, \\ \mathbb{F}_2\alpha & \text{if } q = 1, \\ \mathbb{F}_2(\alpha \smile \alpha) & \text{if } q = 2, \\ 0 & \text{otherwise.} \end{cases}$$

2.4.3. **The cohomology ring of a space.** We have now shown that the cohomology group $H^\bullet(X; R)$ forms a $\mathbb{Z}$*-graded* R*-module* of the form $H^\bullet(X; R) = \bigoplus_{q \in \mathbb{Z}} H^q(X; R)$ and an element α of the component $H^q(X; R)$ is called a *homogeneous element of degree* q, which we indicate by writing $|\alpha| = q$. Equipping this $\mathbb{Z}$-graded R-module with multiplication given by the cup product, that is,

$$\begin{aligned} \smile \, : H^\bullet(X; R) \otimes_R H^\bullet(X; R) &\longrightarrow H^\bullet(X; R), \\ \left(\sum_i \alpha_i \right) \otimes \left(\sum_j \beta_j \right) &\longmapsto \sum_{i,j} \alpha_i \beta_j \coloneqq \sum_{i,j} \alpha_i \smile \beta_j, \end{aligned}$$

makes $H^\bullet(X; R)$ into an algebra satisfying $|\alpha\beta| = |\alpha| + |\beta|$ for all $\alpha, \beta \in H^\bullet(X; R)$. This is to say $H^\bullet(X; R)$ is a $\mathbb{Z}$*-graded* R*-algebra*, and we know from earlier that it is also graded commutative.

The *cohomology ring* of a topological space X with coefficients in a ring R is the $\mathbb{Z}$-graded

R-algebra (and hence a ring) $H^\bullet(X;R)$ with multiplication given by the cup product and the cohomology class $[\varepsilon]$ of the augmentation map $\varepsilon\colon \sigma \mapsto 1_R$ as its unit. The cohomology ring $H^\bullet(X;R)$ is graded commutative in the sense that if $\alpha, \beta \in H^\bullet(X;R)$ then $\alpha\beta = (-1_R)^{|\alpha||\beta|}\beta\alpha$.

From our example above, it is now clear that $H^\bullet(\mathbb{RP}^2;\mathbb{F}_2) \cong \mathbb{F}_2[\alpha]/(\alpha^3)$ as graded rings. By $\mathbb{F}_2[\alpha]$ we mean the one-variable polynomial algebra over $\mathbb{F}_2$ generated by α with $|\alpha| = 1$, whose graded structure is given by $(\mathbb{F}_2[\alpha])^q = \delta_{q>0}\mathbb{F}_2\alpha^q$. And from the example preceding it, we know $H^\bullet(\mathbb{RP}^2\#\mathbb{RP}^2;\mathbb{F}_2)$ is generated by α_1, α_2 and is completely determined by the relations $|\alpha_1| = |\alpha_2| = 1$, $\alpha_1\alpha_2 = 0$, and no nonzero element has degree ≥ 3. Thus, as graded rings, $H^\bullet(\mathbb{RP}^2\#\mathbb{RP}^2;\mathbb{F}_2) \cong \mathbb{F}_2[\alpha_1,\alpha_2]/(\alpha_1\alpha_2, \alpha_1^2 - \alpha_2^2, \dots)$, where the "$\cdots$" indicate quotienting out to adhere to the last of these properties.

2.4.4. **Borsuk–Ulam theorem with the cup product.**

Exercise 2.4.4.1.

(a) Using the cup product structure, show there is no map $g : \mathbb{RP}^n \to \mathbb{RP}^m$ inducing a nontrivial map $H^1(\mathbb{RP}^m;\mathbb{Z}/2\mathbb{Z}) \to H^1(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$ if $n > m$.

(b) Prove the Borsuk–Ulam theorem according to which for every map $f : S^n \to \mathbb{R}^n$ there exists a point $x \in S^n$ with $f(x) = f(-x)$.

Solution. (a) Suppose $n > m$ and the map $g : \mathbb{RP}^n \to \mathbb{RP}^m$ induces a nontrivial homomorphism $g^* : H^1(\mathbb{RP}^m;\mathbb{F}_2) \to H^1(\mathbb{RP}^{m+k};\mathbb{F}_2)$. Identifying g^* as a map between $\mathbb{Z}$-graded $\mathbb{F}_2$-algebras, we obtain a nontrivial $\mathbb{F}_2$-module homomorphism $g^* : \mathbb{F}_2[\beta]/(\beta^{m+1}) \to \mathbb{F}_2[\alpha]/(\alpha^{n+1})$, where $|\alpha| = |\beta| = 1$. Since g^* preserves cup products and their degrees, we can identify g^* as a nontrivial map of $\mathbb{Z}$-graded $\mathbb{F}_2$-algebras. Hence g^* then sends elements of degree 1 to element of degree 1, so since the only non-trivial elements of degree 1 in $\mathbb{F}_2[\alpha]/(\beta^{n+1})$ is α, we must have $g^*\beta = \alpha$ since g^* is nontrivial. Since the order of α is larger than m, we can write

$$0 \neq \alpha^m = g^*(\beta^m) = g^*(0) = 0,$$

a contradiction, so we must have $g^*\alpha = 0$. Hence g^* is trivial.

(b) Suppose toward a contradiction $f : S^n \to \mathbb{R}^n$ satisfies $f(x) \neq f(-x)$ for all $x \in S^n$. Then $g : S^n \to S^{n-1}$ given by

$$g(x) \coloneqq \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$$

is a well-defined and continuous map satisfying $g(-x) = -g(x)$ for all $x \in S^n$, so g respects the quotient map $S^{n-1} \twoheadrightarrow \mathbb{RP}^{n-1}$. Thus g descends to a map $\overline{g} : \mathbb{RP}^n \to \mathbb{RP}^{n-1}$.

If $n = 1$, then $g : S^1 \to S^0 = \{\pm 1\}$ cannot satisfy $g(-x) = -g(x)$ since the fact $\{\pm 1\}$ is disconnected implies g is constant, so assume $n \geq 2$. By part (a), the induced map $\overline{g}^* : H^1(\mathbb{RP}^{n-1}; \mathbb{F}_2) \to H^1(\mathbb{RP}^{n-1}; \mathbb{F}_2)$ is the trivial map, that is, that $\overline{g}^* \alpha = 0$ for each $\alpha \in H^1(\mathbb{RP}^{n-1}; \mathbb{F}_2)$.
We now show that $\overline{g}^*$ is also nontrivial, which will give the result. By the universal coefficient theorem and the Hurewicz isomorphism, we have

$$H^1(\mathbb{RP}^{n-1}; \mathbb{F}_2) \cong H_1(\mathbb{RP}^{n-1}; \mathbb{F}_2) \cong \pi_1^{\mathrm{ab}}(\mathbb{RP}^{n-1}) \cong (\mathbb{F}_2)^{\mathrm{ab}} \cong \mathbb{F}_2.$$

It follows that there exists a nontrivial element $\alpha \in H^1(\mathbb{RP}^{n-1}; \mathbb{F}_2)$, which corresponds to the nontrivial based loop α in $\pi_1^{\mathrm{ab}}(\mathbb{RP}^{n-1})$. Since the projection map $S^{n-1} \twoheadrightarrow \mathbb{RP}^{n-1}$ is a double covering map, α lifts to a path joining two antipodal points in S^{n-1}. Since $g(x) = g(-x)$ for each $x \in S^{n-1}$, we know g preserves antipodes; Hence the lift under the double cover $S^n \twoheadrightarrow \mathbb{RP}^n$ of the image of α under g is also map joining two antipodes, which then corresponds precisely to the nontrivial element in $\pi_1(\mathbb{RP}^n)$. But we just showed that a nonzero element α maps under g^* to a nontrivial element, contradicting g^* is trivial. This completes the proof of the Borsuk–Ulam theorem. □

2.5. POINCARÉ DUALITY

2.5.1. Poincaré duality with $\mathbb{F}_2$-coefficients.

Here we write $\mathbb{F}_2$ for the field $\mathbb{Z}/(2)$.

Theorem 2.5.1.1. *If M is a connected closed n-manifold, then there is an isomorphism*

$$\mathrm{PD}\colon H^k(M; \mathbb{F}_2) \xrightarrow{\cong} H_{n-k}(M; \mathbb{F}_2).$$

In particular, where $[M]$ is the generator of $H^0(M; \mathbb{F}_2) \cong \mathbb{F}_2$, we have

$$\begin{aligned} \mathbb{F}_2 \cong H^0(M; \mathbb{F}_2) &\xrightarrow{\mathrm{PD}} H_n(M; \mathbb{F}_2), \\ 1 &\longmapsto [M]. \end{aligned}$$

Moreover, for all $\alpha \in H^{n-k}(M; \mathbb{F}_2)$ and $b \in H_{n-k}(M; \mathbb{F}_2)$,

$$\langle \alpha, b \rangle = \left\langle \alpha \smile \overset{-1}{\mathrm{PD}}(b), [M] \right\rangle.$$

Here $[M]$ is called the *fundamental class.*

Example 2.5.1.2. Consider S^1, which we think of as a point with a single loop a. Then $a \in Z_1(S^1; \mathbb{F}_2)$, and $[a] = [S^1] \in H_1(S^1; \mathbb{F}_2)$.

Now let V be a closed codimension k submanifold of a connected closed n-manifold M.

Consider the following mappings.

$$\begin{tikzcd}
H_{n-k}(V;\mathbb{F}_2) \arrow[r, "i_*^V"] & H_{n-k}(M;\mathbb{F}_2) \arrow[r, "\mathrm{PD}^{-1}", "\cong"'] & H^k(M;\mathbb{F}_2) \\[-20pt]
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\[-20pt]
{[V]} \arrow[r, mapsto] & {[V] \overset{\substack{\text{abusing}\\ \text{notation}}}{=} i_*^V[V]} \arrow[r, mapsto] & \mathrm{PD}^{-1}([V])
\end{tikzcd}$$

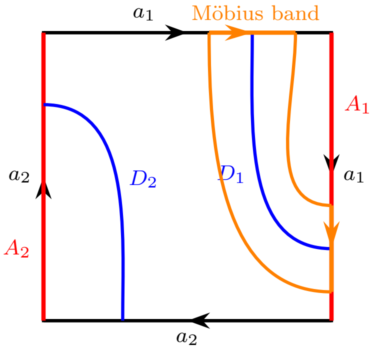
Definition 2.5.1.3. Let M be a connected closed smooth n-manifold. We say closed smooth submanifolds V and W of M are *transverse* to each other if each $x \in V \cap W$, $T_xV + T_xW = T_xM$.

Note that in the notation of the above definition, if V and W are of codimensions k and ℓ in M, respectively, and are transverse, then $V \cap W \subseteq M$ is a closed smooth submanifold of codimension $k + \ell$.

Theorem 2.5.1.4. *Let M be a connected closed* smooth *n-manifold, let V and W be smooth submanifolds of codimensions k and ℓ, respectively, and suppose V and W are* transverse *to each other. Then*

$$\begin{array}{ccccc}
\mathrm{PD}^{-1}([V \cap W]) & = & \mathrm{PD}^{-1}([V]) & \smile & \mathrm{PD}^{-1}([W]). \\
\rotatebox{-90}{$\in$} & & \rotatebox{-90}{$\in$} & & \rotatebox{-90}{$\in$} \\
H_{n-(k+\ell)}(M;\mathbb{F}_2) & & H^k(M;\mathbb{F}_2) & & H^\ell(M;\mathbb{F}_2)
\end{array}$$

Example 2.5.1.5. We will demonstrate the usefulness of Poincaré duality by using it to redo a previous example. Again consider $X_2 = \mathbb{RP}^2 \# \mathbb{RP}^2$, this time as the following diagram.



Here $D_1, D_2 \subseteq X_2$ are smooth submanifolds of codimension 1, and correspond to cohomology classes $\alpha_i \coloneqq \mathrm{PD}^{-1}([D_i]) \in H^1(X_2;\mathbb{F}_2)$.

Claim 2.5.1.6. *$\{\alpha_1, \alpha_2\}$ is a basis for $H^1(X_2;\mathbb{F}_2)$ dual to the basis $\{a_1, a_2\}$ of $H_1(X_2;\mathbb{F}_2)$.*

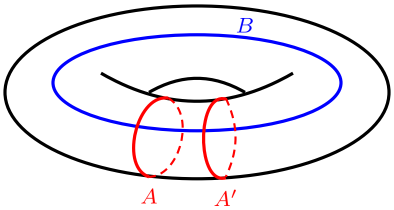
Proof. Let $a_i = [A_i]$, where A_i is in the diagram above. We claim $\langle \alpha_i, a_j\rangle = \delta_{ij}$. We have

$$\langle \alpha_i, a_j\rangle = \left\langle \overset{-1}{\mathrm{PD}}([D_i]), [A_j]\right\rangle \overset{\text{(PD)}}{=} \left\langle \overset{-1}{\mathrm{PD}}([D_i]) \smile \overset{-1}{\mathrm{PD}}([A_j]), [X_2]\right\rangle \overset{(2.5.1.4)}{=} \left\langle \overset{-1}{\mathrm{PD}}[D_i \cap A_j]), [X_2]\right\rangle$$
$$= \left\langle 1 \smile \overset{-1}{\mathrm{PD}}([D_i \cap A_j]), [X_2]\right\rangle \overset{\text{(PD)}}{=} \langle 1, [D_i \cap A_j]\rangle = \delta_{ij}.$$

Applying the theorem again, we obtain $\alpha_1 \smile \alpha_2 = \mathrm{PD}^{-1}([D_1]) \smile \mathrm{PD}^{-1}([D_2]) = \mathrm{PD}([D_1 \cap D_2]) = 0$ and $\alpha_1 \smile \alpha_1 = \mathrm{PD}^{-1}([D_1]) \smile \mathrm{PD}^{-1}([D_1]) = \mathrm{PD}^{-1}([D_1]) \smile \mathrm{PD}^{-1}([A_1]) = \mathrm{PD}^{-1}([D_1 \cap A_1]) \neq 0$, so $\{\alpha_1, \alpha_2\}$ is dual to $\{a_1, a_2\}$. ☐

This agrees with the same result we found in the previous example.

Example 2.5.1.7. Let T be the torus, which we consider as the following diagram.



Then $H_1(T;\mathbb{F}_2) = \mathbb{F}_2[A] \oplus \mathbb{F}_2[B]$. By Poincaré duality, we know $H^1(T;\mathbb{F}_2) \cong \mathbb{F}_2\alpha \oplus \mathbb{F}_2\beta$, where $\alpha = \mathrm{PD}^{-1}([A])$ and $\beta = \mathrm{PD}^{-1}([B])$. Then $\alpha \smile \beta = \mathrm{PD}^{-1}([A]) \smile \mathrm{PD}^{-1}([B]) \overset{(2.5.1.4)}{=} \mathrm{PD}^{-1}([A \cap B]) \neq 0 \in H^2(T;\mathbb{F}_2)$. To apply the theorem, we need transversality. If we use $[A]$ twice, we cannot do this. To avoid this issue, homotope $[A]$ to $[A']$, as in the picture, so that by homotopy invariance the homology class is the same. The intersection $A \cap A'$ is empty, so vacuously the intersection is transversal. Applying Theorem 2.5.1.4, we obtain

$$\alpha \smile \alpha = \overset{-1}{\mathrm{PD}}([A]) \smile \overset{-1}{\mathrm{PD}}([A']) = \overset{-1}{\mathrm{PD}}([A \cap A']) = 0,$$

and $\beta \smile \beta = 0$ is similar. It follows that there exists a $\mathbb{Z}$-graded $\mathbb{F}_2$-algebra isomorphism $H^\bullet(T;\mathbb{F}_2) \xleftarrow{\cong} \mathbb{F}_2[\alpha,\beta]/(\alpha^2,\beta^2)$.

2.5.2. **Orientations on non-smooth manifolds.** Recall that if M is a closed topological manifold of dimension 2, then $H_2(M;\mathbb{Z})$ is $\mathbb{Z}$ if M is orientable and 0 otherwise. Further recall that if M is a *smooth* n-manifold, then we defined an *orientation* on M to be a collection of maps $\{M \ni x \mapsto o_x \mid x \in M\}$, where o_x is an orientation on T_xM, that is locally consistent in the sense of local trivializations of the tangent bundle TM. Equivalently, an orientation is a section σ of the *orientation cover*, which is the fiber bundle $\pi\colon \widetilde{M} \to M$ where $\widetilde{M} = \{(x, o_x) \mid x \in M,\ o_x \text{ is an orientation of the vector space } T_xM\}$ with structure group $\mathbb{F}_2$. The following result is a corollary of the classification of covering spaces via deck transformations.

Corollary 2.5.2.1. *If a smooth manifold M is simply connected or there are no subgroups of index 2 in $\pi_1(M)$, then M is orientable.*

The characterization of the notion of orientation on a smooth manifold, as a section of the orientation cover, easily generalizes to possibly non-smooth topological manifolds. To do this, though, we need an analog to the orientation cover for topological manifolds that does not use the tangent bundle. Luckily we already know of such a notion, which we now recall.

If V an $\mathbb{R}$-vector space of dimension n, then $H_q(V, V \smallsetminus 0) = \delta_{q=n}\mathbb{Z}$. Moreover, there is a correspondence

$$\begin{aligned} \{\text{orientations on } V\} &\longleftrightarrow \{\text{generators of } H_n(V, V \smallsetminus 0)\} \\ [\text{basis } \{b_1, \dots, b_n\}] &\longmapsto [\{b_1, \dots, b_n, -(b_1 + \cdots + b_n)\}]\,. \end{aligned}$$

We used this to define orientation on a topological manifold.

The *orientation cover* of a topological n-manifold X is the bundle $\pi\colon \widetilde{X} \to X$ where $\widetilde{X} = \{(x, \mu_x) \mid x \in X, \mu_x \text{ is a generator of } H_n(M, M \smallsetminus x)\}$ with structure group $\mathbb{Z}^\times \cong \mathbb{F}_2$. An *orientation* on X is a section μ of π.

2.5.3. **Poincaré duality with $\mathbb{Z}$-coefficients.** We can now use the notion of an orientation for a possibly non-smooth topological manifold.

Theorem 2.5.3.1. *For a closed connected orientable topological n-manifold X, there is an isomorphism* PD: $H^k(M) \xrightarrow{\cong} H_{n-k}(M)$ *determined by* $H^0(M) \ni 1 \mapsto [M] \in H_n(M)$ *where* $[M]$ *is the fundamental class. Moreover, for all* $\alpha \in H^{n-k}(M)$ *and* $b \in H_{n-k}(M)$, *we have* $\langle \alpha, b \rangle = \langle \alpha \smile \mathrm{PD}^{-1}(b) \big| [M] \rangle$.

2.5.4. **Cup products of fundamental classes.** Let M be a closed connected smooth oriented n-manifold with fundamental class $[M] \in H_n(M)$. Let V and W be oriented submanifolds of codimensions k and ℓ, respectively. Then $V \cap W$ is a submanifold of codimension $k + \ell$.

Lemma 2.5.4.1. *If* $0 \to A \to B \to C \to 0$ *is a short exact sequence of finite-dimensional real vector spaces, then an orientation on any two of these uniquely determines a compatible orientation on the third.*

Theorem 2.5.4.2. *Let M be a connected closed oriented smooth n-manifold with fundamental class $[M] \in H_n(M)$. Let V and W be oriented submanifolds of codimensions k and ℓ, respectively. If V and W are transverse to each other, then*

$$\begin{array}{ccccc} \mathrm{PD}^{-1}([V]) & \smile & \mathrm{PD}^{-1}([W]) & = & \mathrm{PD}^{-1}([V \cap W]) \\ \rotatebox{90}{$\Subset$} & & \rotatebox{90}{$\Subset$} & & \rotatebox{90}{$\Subset$} \\ H^k(M) & & H^\ell(M) & & H_{n-(k+\ell)}(M) \end{array}$$

Example 2.5.4.3. Complex projective space $\mathbb{CP}^n$ has $2k$-skeleton $\mathbb{CP}^k \subseteq \mathbb{CP}^n$, and $H^{2k}(\mathbb{CP}^n) = \delta_{0\leq k\leq n}\mathbb{Z}$. As in the real case, we get a graded ring isomorphism $H^\bullet(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$ where $\alpha \in H^2$ (i.e., $|\alpha| = 2$).

2.5.5. **Poincaré duality with commutative R-coefficients.**

To formulate Poincaré duality for R-coefficients for any commutative ring R, we need a weaker condition of orientability.

Fix a commutative ring R and let M be any n-manifold. An *R-orientation cover* of M is a principal $R^\times$-bundle $\pi\colon \widetilde{M}_R \to M$ where

$$\widetilde{M}_R = \{(x,\mu_x) \mid x \in M, \mu_x \text{ is a generator of } H_n(M, M \smallsetminus x; R)\}\,.$$

This means group of automorphisms (deck transformations) of the total space is isomorphic to $R^\times$. An *R-orientation* of M is a section μ of π.

Example 2.5.5.1. $\widetilde{M}_\mathbb{Z}$ is the usual orientation cover from before. $\widetilde{M}_{\mathbb{F}_2} \cong M$. $\widetilde{M}_\mathbb{R} = \mathbb{R}^\times$-cover of M, which when M is smooth is precisely $\bigwedge^n T^*M \smallsetminus \{0\}$.

Theorem 2.5.5.2. *If R is a commutative ring and M is a connected closed R-orientable manifold, then an R-orientation induces an isomorphism* $\mathrm{PD}\colon H^k(M;R) \xrightarrow{\cong} H_{n-k}(M;R)$.

2.5.6. **An extra pairing within the cohomology ring.**

Fix a commutative ring R and an R-module M. Recall that by the universal coefficient theorem, there is a split short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_R(H_{n-k-1}(M;R),R) \longrightarrow H^{n-k}(M;R) \xrightarrow{\mathrm{ev}} \mathrm{Hom}_R(H_{n-k}(M;R),R) \longrightarrow 0$$

Further recall that in the case $R = \mathbb{Z}$, the Ext group is torsion, and that when R is a field, then ev is an isomorphism. In the general case, we can consider the mappings

$$\begin{tikzcd} H^{n-k}(M;R) \arrow[r, "\mathrm{ev}"] & \mathrm{Hom}_R(H_{n-k}(M;R),R) \arrow[r, "\mathrm{PD}^\vee"] & \mathrm{Hom}_R(H^k(M;R),R) \\ \alpha \arrow[r, mapsto] & (b \mapsto \langle \alpha, b\rangle) \arrow[r, mapsto] & \left(\left(\beta \xrightarrow{\mathrm{PD}} \mathrm{PD}(\beta)\right) \longmapsto \langle \alpha, \mathrm{PD}(\beta)\rangle\right) \end{tikzcd}$$

Since $\langle \alpha, \mathrm{PD}(\beta)\rangle = \langle \alpha \smile \beta, [M]\rangle$, this gives us a pairing

$$\begin{aligned} H^{n-k}(M;R) \times H^k(M;R) &\longrightarrow R, \\ (\alpha,\beta) &\longmapsto \langle \alpha \smile \beta, [M]\rangle\,. \end{aligned} \tag{2.5.6.1}$$

Let A, B be R-modules. An R-bilinear map $A \times B \xrightarrow{\omega} R$ is called *nonsingular* if the R-module map $A \to \mathrm{Hom}_R(B,R)$ sending $a \in A$ to the map $b \mapsto \omega(a,b)$ is an isomorphism.

Poincaré duality and the UCT now imply the following.

Proposition 2.5.6.2. *If either*

(i) *R is a field, or*

(ii) *$R = \mathbb{Z}$ and we replace $H^\bullet(M;\mathbb{Z})$ by $H^\bullet(M;\mathbb{Z})/Torsion$,* *then the pairing* (2.5.6.1) *is nonsingular, that is,* $H^{n-k}(M;R) \to \operatorname{Hom}_R(H^k(M;R),\mathbb{Z})$ *is an isomorphism.*

Example 2.5.6.3. Again consider $M = \mathbb{CP}^n$.

Claim 2.5.6.4. *We can now say that if α is a generator of $H^2(\mathbb{CP}^n)$ then α^n is a generator of $H^{2n}(\mathbb{CP}^n)$. It follows that we have a* ring *isomorphism*

$$H^\bullet(\mathbb{CP}^n) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1}).$$

Proof. We argue by over $n \geq 2$. If $n = 2$ and α is a generator of $H^2(\mathbb{CP}^n)$ then by nonsingularity of (2.5.6.1) and the proposition we know there exists a $\beta \in H^2(\mathbb{CP}^n)$ such that $\alpha \smile \beta$ is a generator of $H^4(\mathbb{CP}^n)$. Since $H^2(\mathbb{CP}^4) \cong H^4(\mathbb{CP}^4) \cong \mathbb{Z}$, α^2 is a generator of $H^2(\mathbb{CP}^2)$.

For the induction step, by the inductive hypothesis the inclusion map $i\colon \mathbb{CP}^n \hookrightarrow \mathbb{CP}^{n+1}$ induces isomorphisms $i_*\colon H^q(\mathbb{CP}^n) \xrightarrow{\cong} H^q(\mathbb{CP}^{n+1})$ for all $0 \leq q \leq 2n$. Hence, if α_{n+1} generates $H^2(\mathbb{CP}^{n+1})$, then its image $i^*\alpha_{n+1} = \alpha_n$ generates $H^q(\mathbb{CP}^{n+1})$.

By the previous proposition, there exists $\beta = \pm\alpha_{n+1} \in H^2(\mathbb{CP}^{n+1}) \cong \mathbb{Z}$ such that $\alpha_{n+1}^n \smile \beta (= \alpha_{n+1}^{n+1})$ is a generator of $H^{2n+1}(\mathbb{CP}^{n+1})$, so we are done. □

2.5.7. **Künneth theorem for cohomology.** Define the *cohomology cross product* by the pairing

$$\begin{aligned} H^k(X;R) \times H^\ell(Y;R) &\xrightarrow{\times} H^{k+\ell}(X \times Y;R), \\ (\alpha,\beta) &\longmapsto \pi_1^*\alpha \smile \pi_2^*\beta, \end{aligned} \tag{2.5.7.1}$$

where $\pi_1\colon X \times Y \to X$ and $\pi_2\colon X \times Y \to Y$ are the projection maps.

Theorem 2.5.7.2 (Künneth Theorem for Cohomology)**.** *There is a natural short exact sequence of the form*

$$0 \to (H^\bullet(X;R) \otimes_R H^\bullet(Y;R))^q \xrightarrow{\times} H^q(X \times Y;R) \to (\operatorname{Tor}_1^R(H^\bullet(X;R), H^\bullet(Y;R)))^{q-1} \to 0$$

that splits, but not naturally. In particular, $\times$ *is an isomorphism if either*

(i) *R is a field, or*

(ii) *$R = \mathbb{Z}$ and the cohomology is torsion-free in at least one factor.*

Remark 2.5.7.3. The cohomology cross product $H^\bullet(X;R) \otimes_R H^\bullet(Y;R) \xrightarrow{\times} H^\bullet(X \times Y;R)$ is a ring homomorphism and an isomorphism in cases (i) and (ii) above.

Note that if A and B are graded commutative R-algebras, then $A \otimes_R B$ is a graded commutative R-algebra with multiplication given by

$$\begin{aligned} A \otimes_R B \times A \otimes_R B &\longrightarrow A \otimes_R B \\ (a \otimes b, a' \otimes b') &\longmapsto (-1)^{|a'|\cdot|b|} aa' \otimes bb'. \end{aligned}$$

This is very useful, since we can now compute the cohomology rings of product spaces.

Example 2.5.7.4. $H^\bullet(\mathbb{CP}^m \times \mathbb{CP}^n; \mathbb{Z}) \cong H^\bullet(\mathbb{CP}^m) \otimes H^\bullet(\mathbb{CP}^n) \cong \mathbb{Z}[\alpha]/(|a^{m+1}) \otimes \mathbb{Z}[\beta]/(\beta^{n+1}) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^{m+1}, \beta^{n+1})$.

Proposition 2.5.7.5. *For a connected closed n-manifold M, if n is odd, then the Euler characteristic $\chi(M)$ is zero.*

Proof. Let $n = 2m + 1$ for an integer m. Then $\chi(M) = \chi(M; \mathbb{F}_2) = \sum_{i=0}^{n} (-1)^i c_i$, where $c_i = \dim H_i(M; \mathbb{F}_2)$. Then $H_k(M; \mathbb{F}_2) \underset{\text{PD}}{\cong} H^{n-k}(M; \beta_2) \underset{\text{UCT}}{\cong} H_{n-k}(M; \mathbb{F}_2)$, so $c_k = c_{n-k}$. Hence the alternating sum vanishes. □

Exercise 2.5.7.6. The *Kummer surface* is the submanifold of $\mathbb{CP}^3$ of (real) dimension 4 is defined by

$$K \coloneqq \left\{ [z_0, z_1, z_2, z_3] \in \mathbb{CP}^3 \mid z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0 \right\}$$

It can be shown that

- K is simply connected, and
- K has Euler characteristic $\chi(K) = 24$.

Use these facts to calculate the homology groups $H_q(K)$ and the cohomology groups $H^q(K)$ for all q. Hint: Use the universal coefficient theorem and Poincaré duality to relate homology and cohomology groups. Make sure to provide an argument for why the assumptions of the Poincaré duality theorem are satisfied.

Exercise 2.5.7.7. ($\mathbb{RP}^3$ is not homotopy equivalent to $\mathbb{RP}^2 \vee S^3$)

(a) Let X, Y be connected topological spaces equipped with basepoints $x_0 \in X, y_0 \in Y$. Let $X \vee Y$ be their wedge sum, and let $\pi_1 : X \vee Y \to X, \pi_2 : X \vee Y \to Y$ be the natural projection maps. Show that if $\alpha \in H^\bullet(X; R), \beta \in H^\bullet(Y; R)$ with $|\alpha|, |\beta| \geq 1$, then $\pi_1^* \alpha \smile \pi_2^* \beta = 0$.

(b) Use cup products to show that $\mathbb{RP}^3$ is not homotopy equivalent to $\mathbb{RP}^2 \vee S^3$.

2.5.8. **De Rham's theorem: singular cohomology is de Rham cohomology.** Let M be a smooth n-manifold, TM the tangent bundle, and T^*M the cotangent bundle. Recall that the kth exterior power bundle is $\bigwedge^k T^*M$ and $\Omega^k(M) = C^\infty(M, \bigwedge^k T^*M) =$ is the space of smooth sections of $\bigwedge^k T^*M$, i.e., of *differential k-forms* on M. Recall that k-forms can be integrated over compact oriented submanifolds of M. Also $\Omega^\bullet(M) = \bigoplus_k \Omega^k(M)$ has a graded commutative algebra structure with respect to the wedge product $\wedge$ on differential forms. Recall that where d is the de Rham differential, de Rham cohomology is the cohomology $H^\bullet_{\mathrm{dR}}(M)$ given by $H^\bullet(\Omega^\bullet(M), \mathrm{d})$, the cohomology of the de Rham complex $\cdots \to \Omega^{k-1}(M) \xrightarrow{\mathrm{d}} \Omega^k(M) \xrightarrow{\mathrm{d}} \Omega^{k+1}(M) \to \cdots$, and is an $\mathbb{R}$-vector space. One may wonder if

$H^\bullet_{\mathrm{dR}}(M)$ and $H^\bullet(M;\mathbb{R})$ coincide. This is indeed the case in most situations, and we now discover why.

Can we produce a cochain map relating the cochain complex $\Omega^\bullet(M)$ and $C^\bullet(M;\mathbb{R})$? Well, if the singular n-simplices we will ultimately not work with are not smooth, then there may be strange behavior going on, so it would be nice to at least first connect $\Omega^\bullet(M)$ and $C^\bullet_{sm}(M;\mathbb{R})$, where the latter of these is the subcochain complex of $C^\bullet(M;\mathbb{R})$ containing the duals of only smooth singular chains.

Where C_k^{sm} consists of linear combinations of smooth singular k-simplicies, then, we can consider the map $I\colon \Omega^k(M) \to C^k_{sm}(M;\mathbb{R}) =: \mathrm{Hom}(C_k^{sm}(M))$ given by $\omega \mapsto I(\omega)$, where $I(\omega)$ is the map $C_k(M) \to R$ determined by

$$\langle I(\omega)|\sigma\colon \Delta^n \to M\rangle := \int_{\Delta^k} \sigma^*\omega,$$

where we note $\sigma^*\omega \in \Omega^k(\Delta^k)$. (As $\sigma^*\omega$ only makes sense when σ is smooth, this is why we are restricting to duals of smooth singular chains.) To check this map is indeed a cochain map, we need to check that I is compatible with boundary maps, that is, that the diagram

$$\begin{tikzcd}
\Omega^k(M) \arrow[r, "I"] \arrow[d, "\mathrm{d}"'] & C^k_{sm}(M;\mathbb{R}) \arrow[d, "\delta"] \\
\Omega^{k+1}(M) \arrow[r, "I"] & C^{k+1}_{sm}(M;\mathbb{R})
\end{tikzcd}$$

commutes. This can be shown by arguing the bras $\langle \delta I(\omega)|$ and $\langle I(\mathrm{d}\omega)|$ agree. And indeed they do, since for any given $\sigma\colon \Delta^{k+1} \to M$, we have

$$\begin{aligned}
\left\langle \delta I(\omega) \middle| \Delta^{k+1} \xrightarrow{\sigma} M \right\rangle &= \langle I(\omega)|\partial\sigma\rangle = \sum_{i=0}^{k+1} (-1)^i \left\langle I(\omega)|\sigma \circ [v_0, \dots, \widehat{v}_i, \dots, v_{k+1}]\right\rangle \\
&= \sum_{i=0}^{k+1} (-1)^i \int_{\Delta^k} [v_0, \dots, \widehat{v}_i, \dots, v_{k+1}]^* \sigma^*\omega \overset{\star}{=} \int_{\partial\Delta^{k+1}} \sigma^*\omega \\
&= \int_{\Delta^{k+1}} \delta(\sigma^*\omega) = \int_{\Delta^{k+1}} \sigma^*(\delta\omega) = \langle I(\mathrm{d}\omega)|\sigma\rangle,
\end{aligned}$$

where the starred equality is by Stokes' theorem. This gives us a cochain map $\Omega^\bullet(M) \to C^\bullet_{sm}(M;\mathbb{R})$. We will not show this, but it turns out this map is indeed an isomorphism.

Chapter 3

More Homotopy Theory

3.1. COFIBRATIONS

3.1.1. **Homotopy extension property.** A map $(X, A) \in \mathcal{T}_{(2)}$ has the *homotopy extension property* (HEP) if homotopies of maps out of A can be extended to homotopies out of X. More precisely, this means that is a cofibration if for any map $f\colon A \to \mathrm{Path}(Y)$ (thought of as a homotopy) and any map $\widetilde{f}_0\colon X \to Y$ with $\mathrm{ev}_0 \circ f = \widetilde{f}_0 \circ i$, there is a map $\widetilde{f}\colon X \to \mathrm{Path}(Y)$ such that the following diagram on the right commutes, or equivalently, by replacing each map by its currying, if the diagram on the left commutes.

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, hook] & X \arrow[d, hook] \arrow[ddr, bend left, "\widetilde{f}_0"] & \\
\mathrm{Cyl}(A) \arrow[r, "\mathrm{Cyl}(i)"] \arrow[drr, bend right, "f"'] & \mathrm{Cyl}(X) \arrow[dr, dashed, "\exists\, \widetilde{f}"] & \\
 & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"] & \mathrm{Path}(Y) \arrow[d, "\mathrm{ev}_0"] \\
X \arrow[r, "\widetilde{f}_0"'] \arrow[ur, dashed, "\exists\, \widetilde{f}"] & Y
\end{tikzcd}$$

More generally, we call *any* map $i \in \mathcal{T}(A \to X)$ a *cofibration* if it has this property.

We call $X \in \mathcal{T}_*$ *well-pointed* (or *nondegenerately based*) if the pointed map $* \to X$ is a *pointed cofibration*, i.e., a cofibration in $\mathcal{T}_*$.

The diagram in the definition of cofibration is suggestive of a pushout, and this suspicion is rightfully so as the next result shows.

Proposition 3.1.1.1. *A map $i \in \mathcal{T}(A \to X)$ is a cofibration if and only if there is a solution r in $\mathcal{T}$ to the problem*

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, hook] & X \arrow[d, hook] \arrow[ddr, bend left] & \\
\mathrm{Cyl}(A) \arrow[r, "\mathrm{Cyl}(i)"] \arrow[drr, bend right] & \mathrm{Cyl}(X) \arrow[dr, dashed, "\exists\, r"] & \\
 & & M_i
\end{tikzcd}$$

where the outer square is a pushout diagram in $\mathcal{T}$. We call the pushout M_i in the outer square of the following diagram is called the mapping cylinder *of i.*

Proof. If $i \in \mathcal{T}(A \to X)$ is a cofibration, then by the HEP there exists a solution. Conversely, suppose such an extension r exists and we are given maps $h\colon \mathrm{Cyl}(A) \to Y$ and $f\colon X \to Y$. Since the outermost square of the diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, hook] & X \arrow[d, hook] \arrow[ddr, bend left] \arrow[dddrr, bend left, "f"] & & \\
\mathrm{Cyl}(A) \arrow[r, "\mathrm{Cyl}(i)"] \arrow[drr, bend right] \arrow[ddrrr, bend right, "h"'] & \mathrm{Cyl}(X) \arrow[dr, dashed, "r"] & & \\
& & M_i \arrow[dr, dashed, "\exists!\varphi"] & \\
& & & Y
\end{tikzcd}$$

commutes, by the universal property of the pushout there is a unique φ that makes the whole diagram commute. Thus $\widetilde{h} \coloneqq \varphi \circ r$ is an extension of h. ☐

The following proposition is false without the WH condition in the definition of $\mathcal{T}$.

Proposition 3.1.1.2. *In $\mathcal{T}$, a cofibration $i\colon A \to X$ is an inclusion map with closed image.*

Proof. Any extension r as in the above diagram admits a section $j\colon M_i(= \mathrm{Cyl}(A) \cup_i X) \to \mathrm{Cyl}(X)$ given on $\mathrm{Cyl}(A)$ by $\mathrm{Cyl}(i)$ and on X by the inclusion, so the claim follows from elementary point-set topology. ☐

3.1.2. **Constructing cofibrations.** There are several ways to construct cofibrations in $\mathcal{T}$.

Proposition 3.1.2.1.

(i) *In $\mathcal{T}$, the unique map $\varnothing \hookrightarrow X$ is a cofibration.*

(ii) *Cofibrations in $\mathcal{T}_\circ$ are cofibrations in $\mathcal{T}_*$, and the converse is true when the source and target are well-pointed.*

(iii) *In $\mathcal{T}$, homeomorphisms are cofibrations.*

(iv) *In $\mathcal{T}$, the composition of cofibrations is a cofibration.*

(v) *In $\mathcal{T}$, if $f\colon A \to X$ and $g\colon B \to Y$ are cofibrations, then $f \amalg g\colon A \amalg B \to X \amalg Y$ is a cofibration.*

(vi) *In $\mathcal{T}$, pushouts of cofibrations are cofibrations in the sense that if*

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\
B \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & P
\end{tikzcd}$$

is a pushout square and i is a cofibration, then the map $B \to P$ is a cofibration. Moreover, in this case the induced map $\overline{g}\colon X/A \to P/B$ is a homeomorphism.

Proof. Most of these items follow directly from the definitions, but we will show pushouts preserve cofibrations. Suppose we are given that the solid subdiagram in

$$\begin{tikzcd}
A \arrow[r, "H_A"] \arrow[d, "i"] & B \arrow[r, "H_B"] \arrow[d, "j"] & Z^I \arrow[d, "\mathrm{ev}_0"] \\
X \arrow[r, "f"] \arrow[urr, dashed, "\exists H"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

commutes in $\mathcal{T}_\circ$, where $i\colon A \to X$ is a cofibration and the left square is a pushout diagram. Since i is a cofibration, there exists a map $H\colon X \to Z^I$ making the whole diagram commute. But then the solid subdiagram in

$$\begin{tikzcd}
A \arrow[r, "H_A"] \arrow[d, "i"] & B \arrow[d, "\mathrm{ev}_0"] \arrow[ddr, bend left, "H_B"] & \\
X \arrow[r, "f"] \arrow[urr, phantom, ""] \arrow[drr, bend right, "H"'] & Y \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\exists!\widetilde{H}"] & \\
 & & Z^I
\end{tikzcd} \tag{3.1.2.2}$$

commutes, so by the universal property of the pushout there exists a unique map $\widetilde{H}\colon Y \to Z^I$ making the whole diagram commute. Now the solid subdiagram obtained from the above diagram by replacing H with $\widetilde{H} \circ f$ commutes, so there is a unique map $\alpha\colon Y \to \mathrm{Path}(Z)$ making the whole diagram commute. But any such map also makes (3.1.2.2) commute, and $\widetilde{H}$ is the unique map with this property, so $\alpha = \widetilde{H}$, which proves $\widetilde{H}$ is the desired extension. □

Corollary 3.1.2.3. *For any map $f\colon X \to Y$ in $\mathcal{T}$, the inclusion $i\colon Y \hookrightarrow M_f$ is a cofibration.*

Proof. We have a pushout diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, hook, "i"] \\
\mathrm{Cyl}(X) \arrow[r] & M_f \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

and the inclusion $X \hookrightarrow \mathrm{Cyl}(X)$ is a cofibration as an inclusion map with closed image by Proposition 3.1.1.2. □

3.1.3. **Converting a map to a cofibration.** Here we prove the useful fact that, up to equivalence, all maps are cofibrations. In particular, any map can be replaced by a cofibration by changing the target up to homotopy equivalence.

Lemma 3.1.3.1. *For any map $f\colon X \to Y$ in $\mathcal{T}_\circ$, the inclusion $X \hookrightarrow M_f$ into the mapping cylinder is a cofibration. If moreover X is well-pointed, then this is true in $\mathcal{T}_*$.*

Proof. We prove the unpointed case here and leave the case in $\mathcal{T}_*$ when X is well-pointed is left to the reader.

Let $j\colon X \hookrightarrow M_f$ be the inclusion $x \mapsto [(x,0)]$ of X into the mapping cylinder M_f. If the solid subdiagram of

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[d, hook] & M_f \arrow[d, hook] \arrow[rdd, bend left, "g"] & \\
X \times I \arrow[r, "j\times \mathrm{id}_I"] \arrow[rrd, bend right, "H_\bullet"] & M_f \times I \arrow[rd, dashed, "\widetilde{H}_\bullet"] & \\
 & & Z
\end{tikzcd}$$

commutes in $\mathcal{T}_\circ$, i.e., if $g \circ j = H_0$, then the map $\widetilde{H}_\bullet\colon M_f \times I \to Z$ given by $\widetilde{H}_s((x,t)) = H_s(x)$ and $\widetilde{H}_s(y) = g(y)$ makes the whole diagram commute. This map is well-defined, since the identification $((x,1),s) \sim (f(x),s)$ implies

$$\widetilde{H}_s(x,1) = H_s(x) = g(j(x)) = g((x,1)) = g(f(x)) = \widetilde{H}_s(f(x)).$$

And $\widetilde{H}_\bullet$ is continuous again by the pasting lemma. Thus j is a cofibration. ☐

Proposition 3.1.3.2. *All maps $f\colon X \to Y$ in $\mathcal{T}_\circ$ factor as a composition of a cofibration with a homotopy equivalence in a functorial way.*

Proof. Fix a map $f\colon X \to Y$ in $\mathcal{T}_\circ$. Consider $j\colon X \hookrightarrow M_f$ and $q\colon M_f \to Y$ as in the following diagram.

$$\begin{tikzcd}
X \arrow[d, "x\mapsto(x,0)"', "j"] \arrow[rd, "f"] & \\
M_f \arrow[r, "q", "{\substack{(x,t)\mapsto f(x)\\ y\mapsto y}}"'] & Y
\end{tikzcd}$$

Note q is well-defined, since we have $[(x,1)] = f(y)$ and indeed $q([x,1]) = f(x) = q(f(x))$, and is moreover q is continuous by the pasting lemma. By Lemma 3.1.3.1 j is a cofibration, so it only remains to show q is a homotopy equivalence. And indeed, q has homotopy inverse $u\colon Y \to M_f$ given by $u(y) = y$, since $u \circ q(x,t) = u(f(x)) = f(x)$ and $u \circ q(y) = u(y) = y$, while $q \circ u(y) = q(y) = y$. Thus $q \circ u = \mathrm{id}_Y$ and $u \circ q \simeq \mathrm{id}_{M_f}$ via $\widetilde{H}_\bullet\colon M_f \times I \to M_f$ given by $((x,t),s) \mapsto (x, s + t(1-s))$ and $y \mapsto y$. We leave functoriality to the reader. ☐

Exercise 3.1.3.3. State the meaning of functoriality in Proposition 3.1.3.2 and prove it.

3.1.4. **Cofibrations as NDR-pairs.** Fix a space $X \in \mathcal{T}$ and consider a subspace A of X. We call (X,A) a *deformation retract pair (DR-pair)* if there is a map $h\colon \mathrm{Cyl}(X) \to X$ such that

(DR1) $h(x,0) = x$ for all $x \in X$ (i.e., moving points for 0 seconds does nothing), and

(DR2) $h(a,t) = a$ for all $a \in A$ and $t \in I$ (i.e., nothing should be done to points in A), and

(DR3) $h(x,1) \in A$ for all $x \in X$ (i.e., the all points in X end up in A).

We call (X, A) a *neighborhood deformation retract pair (NDR-pair)* if there is a map $\tau\colon X \to I$ and a map $h\colon \operatorname{Cyl}(X) \to X$ satisfying (DR1), (DR2), and moreover

(NDR1) $h(x,t) \in A$ for all $x \in X$ and $t > \tau(x)$ (i.e., if you move a point in the direction of A for the amount of time it would take for that point to reach A, then the result is in A), and

(NDR2) $\tau(a) = 0$ for all $a \in A$ (i.e., points in A take 0 time to get to A).

The "neighborhood" in the name NDR-pair is suggested by the fact that (NDR1) implies $h(x,1) \in A$ for all $x \in U \coloneqq \{x \in X \mid \tau(x) < 1\}$. As τ is continuous, $\tau^{-1}([0,1))$ is an open set containing A, so we have an open neighborhood U of A that deformation retracts onto A. Conversely, if there is an open neighborhood U of A that deformation retracts onto A, say by $h\colon \operatorname{Cyl}(U) \to U$, then we can define $\tau\colon \operatorname{Cyl}(X) \to X$ by $\tau(x) \coloneqq \sup\{t \in I \mid h(x,t) \notin A\}$ and define $\widetilde{h}\colon \operatorname{Cyl}(X) \to X$ by

$$\widetilde{h}(x,t) \coloneqq \begin{cases} 1 & \text{if } (x,t) \notin \operatorname{Cyl}(U), \\ h(x,t) & \text{otherwise.} \end{cases}$$

Theorem 3.1.4.1 ([May11, p. 45])**.** *The following are equivalent for a* closed *subspace A of $X \in \mathcal{T}$.*

(i) *The inclusion $i\colon A \to X$ is a cofibration.*

(ii) *$(\operatorname{Cyl}(X), X \cup \operatorname{Cyl}(A))$ is a DR-pair.*

(iii) *$X \cup_i \operatorname{Cyl}(A)$ is a retract of $\operatorname{Cyl}(X)$.*

(iv) *(X, A) is an NDR-pair.*

3.1.5. **Mapping cones and cofiber sequences.** For $X \in \mathcal{T}_*$, define the *cone* over X to be $CX \coloneqq X \wedge I$. For a pointed map $f\colon X \to Y$, define the *mapping cone* Cf by $Cf \coloneqq Y \cup_f CX$.

Proposition 3.1.5.1 ([Str11, Problem 5.113])**.** *In $\mathcal{T}_*$, we have the following pushout squares.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "\iota_0"'] & Y \arrow[d] \\ CX \arrow[r] & Cf \arrow[from=1-1, phantom, "\ulcorner", very near end] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "j"] \arrow[d, hook, "\iota_0"'] & M_f \arrow[d] \\ CX \arrow[r] & Cf \arrow[from=1-1, phantom, "\ulcorner", very near end] \end{tikzcd}$$

Proof. The square on the left is a pushout square by definition of Cf. The square on the right is a pushout square where $M_f \to Cf$ and $CX \to Cf$ are defined by $[(x,t)] \mapsto [(x,t)]$, $[y] \mapsto [y]$ and $[(x,1)] \mapsto [f(x)]$, $[(x,t)] \mapsto [(x,t)]$ respectively. This makes the right square commute. If $W \in \mathcal{T}_*$ and maps $CX \to W$, $M_f \to W$ also make the diagram commute, then the map $\varphi\colon Cf \to W$ given by $\varphi(a,t) = \beta(f(a),t)$ and $\varphi(y) = *_W$ for $y \in Y \setminus X$ is the unique comparison map. □

Since pushouts satisfy universal properties, by Proposition 3.1.5.1 there is a unique homeomorphism witnessing $Cf = Y \cup_f CX \cong M_f/\overline{j(X)}$. In particular, since Proposition 3.1.3.2 shows that f factors through a cofibration and a homotopy equivalence, while Proposition 3.1.1.2 shows cofibrations are inclusions of closed sets, so every sequence $X \xrightarrow{f} Y \to Cf$ in $\mathcal{T}$ with $Y \to M_f$ the induced map is homotopic to a sequence of the form $A \overset{i}{\hookrightarrow} B \overset{q}{\twoheadrightarrow} B/A$ where i is the inclusion of a closed subset A into B and q is the quotient map. This justifies the sometimes more common terminology *homotopy cofiber* for the mapping cone Cf.

A *strict cofiber sequence* is an inclusion of a (closed) space followed by its quotient map, or equivalently a cofibration followed by the canonical quotient map. Any pointed map $f\colon X \to Y$ induces a strict cofiber sequence $X \overset{j}{\hookrightarrow} M_f \twoheadrightarrow M_f/j(X)$ which is pointwise homotopic to the cofiber sequence $X \xrightarrow{f} Y \to Cf$. We thus call the sequence $X \to Y \to Cf$ the *homotopy cofiber sequence* generated by f. Of course, this is important because strict cofiber sequences are exact sequences of pointed spaces, which are useful for doing computations. The pointwise homotopy equivalence between these sequences, together with the following lemma, thus give for every pointed map $f\colon X \to Y$ an exact sequence of pointed sets $[X,Z] \xleftarrow{f^*} [Y,Z] \xleftarrow{q_f^*} [Cf,Z]$.

Lemma 3.1.5.2. *Let $i\colon A \to X$ be a cofibration in $\mathcal{T}_*$ and let $q\colon X \to X/A$ be the canonical quotient map. Then for any space $Y \in \mathcal{T}_*$, the sequence $[A,Y] \xleftarrow{i^*} [X,Y] \xleftarrow{q^*} [X/A,Y]$ of pointed sets is exact.*

Proof. First observe $i^*q^* = [*]$, since for all $f\colon X/A \to Y$ we have $i^*q^*[f] = [fqi] = [*]$, where the last equality is because $fqi\colon A \to X \to X/A \to Y$ factors through the composite $A \to X \to X/A = *$. Conversely, suppose $f \circ i \simeq *$ for $f\colon X \to Y$. Now replace f with a map such that this is strict equality. We want $g\colon X/A \to Y$ such that $f \circ q \simeq g$. Since $f \circ i = *$, f is constant on A. Thus there exists $\overline{f}\colon X/A \to Y$ such that $\overline{f} \circ q = f$, take $g \coloneqq \overline{f}$. □

Even better, we can extend strict cofiber sequences to "long exact sequences" of pointed spaces in a natural way, which we now show. Fix a pointed map $f\colon X \to Y$ and consider the homotopy cofiber sequence $X \to Y \to Cf$. We can continue this sequence to the right as

$$X \xrightarrow{f} Y \longrightarrow Cf \longrightarrow \Sigma X \xrightarrow{\Sigma f} \Sigma Y \longrightarrow \Sigma Cf \xrightarrow{\Sigma^2 f} \Sigma^2 X \longrightarrow \cdots$$

where the map $Cf \to \Sigma X$ is the map quotienting the mapping cone Cf by Y, which leaves the cone with base collapsed to a point, which is the space ΣX. This sequence is sometimes called a *Puppe sequence* or *long cofiber sequence* associated to f. Applying the functor $[-, Z]\colon \mathcal{T}_* \to \mathsf{Set}_*$ Lemma 3.1.5.2 to it yields the long exact sequence of pointed sets

$$\cdots \longrightarrow [Z, \Sigma^2 X] \xrightarrow{\Sigma^2 f^*} [Z, \Sigma Cf] \longrightarrow [Z, \Sigma Y] \xrightarrow{\Sigma f^*} [Z, \Sigma X] \longrightarrow [Z, Cf] \longrightarrow [Z, Y] \xrightarrow{f^*} [Z, X].$$

3.2. FIBRATIONS

3.2.1. **Homotopy lifting property.** The definition of fibration is formally dual to the definition of cofibration. A map $f\colon E \to B$ satisfies the *homotopy lifting property* if

A map $p\colon E \to B$ is a *fibration* in $\mathcal{T}$ if it satisfies the *homotopy lifting property* for all maps f, which means that for all f, k as in the following solid commutative subdiagram, there exists $\widetilde{k}$ for which the whole diagram commutes.

$$\begin{tikzcd}
X \arrow[rrd, bend left, "k_A"] \arrow[rd, dashed, "\exists\, \widetilde{k}"] \arrow[rdd, bend right, "f"'] & & \\
& \mathrm{Path}(E) \arrow[r, "\mathrm{Path}(i)"] \arrow[d, hook, "\mathrm{ev}_0"] & \mathrm{Path}(B) \arrow[d, hook, "\mathrm{ev}_0"] \\
& E \arrow[r, "p"] & B
\end{tikzcd}$$

Equivalently, a map $p\colon E \to B$ is a fibration if for all f, h as in the following solid commutative diagram, there is a lift $\widetilde{h}$ for which the whole diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\
\mathrm{Cyl}(X) \arrow[r, "h"] \arrow[ru, dashed, "\exists\, \widetilde{h}"] & B
\end{tikzcd}$$

Thus p is a fibration if partial lifts (f) of h can be extended to full lifts $(\tilde{h})$.

When $f\colon E \to B$ is a fibration, we call E and B the *total space* and *base space* respectively, and for $b \in B$ we say the *fiber* of f over b is $f^{-1}(b)$. When B is path-connected, all fibers of a fibration are homotopy equivalent. For $p \in \mathcal{T}(E \to B)$, the *path space* of f is the pullback $E_f = E \times_B \mathrm{Path}(B) = \{(e, \omega) \in E \times \mathrm{Path}(B) \mid \omega(0) = f(e)\}$ in $\mathcal{T}$ as in the outer square of the following diagram.

As for cofibrations, it suffices to consider a "universal test diagram".

Proposition 3.2.1.1. *A map $p \in \mathcal{T}(E \to B)$ is a fibration if and only if there is a solution s in $\mathcal{T}$ to the problem*

$$\begin{tikzcd}
E_p \arrow[rrd, bend left] \arrow[rd, dashed, "\exists\, s"] \arrow[rdd, bend right] & & \\
& \mathrm{Path}(E) \arrow[r, "\mathrm{Path}(p)"] \arrow[d, "\mathrm{ev}_0"] & \mathrm{Path}(B) \arrow[d, "\mathrm{ev}_0"] \\
& E \arrow[r, "p"] & B
\end{tikzcd}$$

where the outer square is a pullback diagram.

Proof. Dualize the proof of Proposition 3.1.1.1. ☐

3.2.2. **Constructing fibrations.** There are several ways to construct fibrations.

Proposition 3.2.2.1.

(i) *In $\mathcal{T}$, maps $X \to *$ are fibrations.*

(ii) *Fibrations in $\mathcal{T}_*$ are fibrations in $\mathcal{T}_\circ$.*

(iii) *In $\mathcal{T}$, homeomorphisms are fibrations.*

(iv) *In $\mathcal{T}$, the composition of fibrations is a fibration.*

(v) *In $\mathcal{T}$, any retract of a fibration is a fibration.*

(vi) *In $\mathcal{T}$, if $f\colon E \to B$ and $g\colon E' \to B'$ are fibrations, then $f \times g\colon E \times E' \to B \times B'$ is a fibration.*

(vii) *In $\mathcal{T}$, pullbacks of fibrations are fibrations in the sense that if*

$$\begin{tikzcd} P \arrow[r] \arrow[d, "q"] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ A \arrow[r] & B \end{tikzcd}$$

is a pullback square and p is a fibration, then q is a cofibration. Moreover, in this case the restriction $p_A\colon E_A := p^{-1}(A) \to A$ is a fibration.

Proof. Dualize the proof of Proposition 3.1.2.1. ☐

3.2.3. **Converting a map to a fibration.** Here we prove the useful fact that, up to pointwise homotopy equivalence, all maps are fibrations. In particular, any map can be replaced by a fibration by changing the domain up to homotopy equivalence.

Lemma 3.2.3.1. *For any map $f\colon X \to Y$ in $\mathcal{T}_\circ$, the map $p\colon E_f \to Y$ is a fibration.*

Proof. Dualize the proof of Lemma 3.1.3.1. Alternatively, see here, Proposition 3.3. ☐

Proposition 3.2.3.2. *All maps $f \in \mathcal{T}_\circ(X \to Y)$ factor as a composition of a homotopy equivalence with a fibration in a functorial way.*

Proof. First observe that the map $r\colon E_f \to X$ given by $r(x, \omega) := \omega(0)$ is a homotopy equivalence with homotopy inverse $i\colon X \to E_f$ given by $x \mapsto (x, c_{f(x)})$, where $c_{f(x)}$ is the $f(x)$-valued constant path in Y; this can be shown by shrinking the path coordinate of E_f down to a constant path. Next define $p\colon X \to E_f$ by $p(x, \omega) := \omega(1)$. By Lemma 3.2.3.1 p is a fibration, and evidently $f = p \circ i$, so we are done.

The functoriality of this construction on $\mathsf{Arr}(\mathcal{T}_\circ)$, i.e., the assertion that "taking the fiber" is a functor $\mathsf{Arr}(\mathcal{T}_\circ) \to \mathcal{T}_\circ$ and the assignment $f \mapsto (Y \to C_f)$ is a natural transformation, is left to the reader.

$\square$

3.2.4. **Path spaces and fiber sequences.** The *strict fiber sequence* of a pointed fibration $p\colon E \to B$ in $\mathcal{T}_*$ is the sequence $F \xrightarrow{i} E \xrightarrow{p} B$ in $\mathcal{T}_*$ where i is the inclusion of the *strict fiber* $F := p^{-1}(*)$ into E. Now fix a pointed map $f\colon X \to Y$ in $\mathcal{T}_*$, and define the *homotopy fiber* of f to be the set $F(f) := \{(x,\omega) \in X \times \mathrm{Path}(Y) \mid \omega(0) = f(x) \text{ and } \omega(1) = *_Y\}$. We call the sequence $F(f) \twoheadrightarrow X \xrightarrow{f} Y$ the *homotopy fiber sequence* of f.

As in the case for cofibrations, the homotopy fiber sequence of a pointed map $f\colon X \to Y$ is pointwise homotopy equivalent to a strict fiber sequence, namely $F(f) \twoheadrightarrow E_f \xrightarrow{p_f} Y$ where $p_f\colon E_f \to Y$ is the fibration constructed in Proposition 3.2.3.2 through which f factors.

By dualizing Lemma 3.1.5.2 and the discussion we had for cofibrations, we obtain for every pointed map $f\colon X \to Y$ an exact sequence of pointed sets $[Z, F(f)] \to [Z,X] \xrightarrow{f_*} [Z,Y]$.

There is a Puppe sequence for f also in this setting, which arises from the observation that the loop space ΩY includes into the mapping fiber $F(f)$, and one can show this extends to obtain

$$\cdots \longrightarrow \Omega^2 Y \longrightarrow \Omega(F(f)) \longrightarrow \Omega X \xrightarrow{\Omega f} \Omega Y \longrightarrow F(f) \longrightarrow X \xrightarrow{f} Y,$$

which is called the *long fiber sequence* associated to f. The long fiber sequence may be easier to work with than the long cofiber sequence, since it deals with subspaces rather than quotient spaces.

Again applying the dual of Lemma 3.1.5.2, we obtain the long exact sequence of pointed sets

$$\cdots \longrightarrow [Z,\Omega^2 Y] \longrightarrow [Z,\Omega(F(f))] \longrightarrow [Z,\Omega X] \xrightarrow{\Omega f_*} [Z,\Omega Y] \longrightarrow [Z,F(f)] \longrightarrow [Z,X] \xrightarrow{f_*} [Z,Y].$$

3.3. (WIP) Factorization theorems and the fundamental lifting property

3.3.1. The unpointed case.

Theorem 3.3.1.1 ([Str11, Theorem 5.42])**.** *In $\mathcal{T}_\circ$, every map $f\colon X \to Y$ factors as the commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "j"'] \arrow[dr, "f"] & \overline{E}_f \arrow[d, "p"] \\
\overline{M}_f \arrow[r, "q"'] & Y
\end{tikzcd}$$

in which

- *i, j are cofibrations,*
- *p, q are fibrations, and*
- *i, q are homotopy equivalences.*

Theorem 3.3.1.2 (The fundamental lifting property)**.** *Suppose that i is a cofibration and p is a fibration in the following commutative diagram in $\mathcal{T}_\circ$.*

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "i"] & E \arrow[d, "p"] \\
X \arrow[r, "f"'] \arrow[ur, dotted] & B
\end{tikzcd}$$

Then the dotted arrow can be filled in to make the diagram strictly commutative if either i or p is a homotopy equivalence.[1]

3.3.2. **The pointed case.** In addition to the following theorem, there is a factorization result for which the well-pointedness hypotheses are dropped; see [Str11, Theorem 5.98].

Theorem 3.3.2.1 ([Str11, Theorem 5.100])**.** *In $\mathcal{T}_*$, every pointed map $f\colon X \to Y$ of well-pointed spaces factors as a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "j"'] \arrow[dr, "f"] & \overline{E}_f \arrow[d, "p"] \\
\overline{M}_f \arrow[r, "q"'] & Y
\end{tikzcd}$$

in which

- *all four spaces are well-pointed;*
- *i, q are pointed (hence unpointed) homotopy equivalences;*
- *i, j are unpointed (hence pointed) cofibrations; and*
- *p, q are pointed (hence unpointed) fibrations.*

Theorem 3.3.2.2 (The pointed fundamental lifting property)**.** *Suppose that i is a pointed cofibration of well-pointed spaces and p is a pointed fibration in the following commutative diagram in $\mathcal{T}_*$.*

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "i"] & E \arrow[d, "p"] \\
X \arrow[r, "f"'] \arrow[ur, dotted] & B
\end{tikzcd}$$

---

[1] In other words, $(C, F \cap W)$ and $(C \cap W, F)$ are weak factorization systems, where C, F, W are the cofibrations, fibrations, and weak equivalences respectively in $\mathcal{T}_\circ$.

Then the dotted arrow can be filled in to make the diagram strictly commutative if either i or p is a pointed homotopy equivalence.

3.4. MODEL CATEGORIES

Here we closely follow [MP12, Chapter 14]. Homotopy theory has two interpretations: the homotopy theory of topological spaces, the core of algebraic topology, and homotopy theory as a general methodology applicable to various subjects. Like category theory, homotopy theory provides a language and substantial results applicable throughout mathematics. This is called *model category theory*, and we study this here.

TODO: add functoriality of the factorization in the definition of model category, if not already there. Brett says this is used for most useful model categories

Let $\mathcal{M}$ be a (locally small) complete and cocomplete category. Note that by considering the coproduct and product of the empty set of objects, this means $\mathcal{M}$ has an initial object $\varnothing$ and a final object $*$.

3.4.1. **Categories with weak equivalences.** A *category with weak equivalences* is a pair $(\mathcal{M}, \mathcal{W})$ where $\mathcal{W}$ is a subcategory of $\mathcal{M}$ satisfying the following conditions.

(i) $\mathcal{W}$ contains all isomorphisms in $\mathcal{M}$,.

(ii) $\mathsf{Arr}(\mathcal{W})$ is *closed under retracts* in $\mathcal{M}$, i.e., if $f \in \mathcal{W}(a \to b)$ and the following diagram commutes in $\mathcal{M}$, then $g \in \mathcal{W}$.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "g"] \arrow[rr, bend left, "\mathrm{id}_A"] & X \arrow[r] \arrow[d, "f"] & A \arrow[d, "g"] \\
B \arrow[r] \arrow[rr, bend right, "\mathrm{id}_B"'] & Y \arrow[r] & B
\end{tikzcd}$$

(iii) $\mathcal{W}$ satisfies the *2-out-of-3 property*, i.e., if any two of the three morphisms u, v, and $u \circ v$ in $\mathcal{M}$ are in $\mathcal{W}$, then so is the third.

We denote morphisms in $\mathcal{M}$ that are in $\mathcal{W}$ by $\overset{\sim}{\to}$. Objects $X, Y \in \mathcal{M}$ are called *weakly equivalent* if there is some nonnegative integer n and weak equivalences as in $X = Z_0 \overset{\sim}{\leftarrow} Z_1 \overset{\sim}{\to} Z_2 \overset{\sim}{\leftarrow} Z_3 \overset{\sim}{\to} Z_4 \overset{\sim}{\leftarrow} \cdots Z_n = Y$ or $X = Z_0 \overset{\sim}{\to} Z_1 \overset{\sim}{\leftarrow} Z_2 \overset{\sim}{\to} Z_3 \overset{\sim}{\leftarrow} Z_4 \overset{\sim}{\to} \cdots Z_n = Y$. This is an equivalence relation, and we will write $X \sim Y$ to mean X and Y are weakly equivalent.

3.4.2. **Lifting properties.** Consider the commutative squares of the following form in $\mathcal{M}$.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"] \arrow[ur, dashed, "\lambda"] & B \end{tikzcd} \tag{3.4.2.1}$$

We say the pair (i, p) has the *lifting property* if for any morphisms f, g making the above square commute, there is a lift λ making both squares commute. For a collection of maps $\mathcal{L}$, we say i has the *right lifting property* (RLP) with respect to $\mathcal{L}$ if (i, p) has the lifting property for all $p \in \mathcal{L}$, and we write $\mathcal{L}^{\boxslash}$ for the collection of all such i. Similarly, for a collection of maps $\mathcal{R}$, we say p has the *left lifting property* (LLP) with respect to $\mathcal{R}$ if (i, p) has the lifting property for all $i \in \mathcal{R}$, and we write ${}^{\boxslash}\mathcal{R}$ for the collection of all such p. We write $\mathcal{L} \boxslash \mathcal{R}$ if (i, p) has the lifting property whenever $i \in \mathcal{L}$ and $p \in \mathcal{R}$, i.e., if $\mathcal{L} \subseteq {}^{\boxslash}\mathcal{R}$, or equivalently if $\mathcal{R} \subseteq {}^{\boxslash}\mathcal{L}$.

3.4.3. **Factoring categories.** An ordered pair $(\mathcal{L}, \mathcal{R})$ of collections of morphisms of $\mathcal{M}$ *factors* $\mathcal{M}$ if every morphism $f \in \mathcal{M}(X \to Y)$ factors as a composite $X \overset{i(f)}{\to} Z(f) \overset{p(f)}{\to} Y$ for some $i(f) \in \mathcal{L}$ and $p(f) \in \mathcal{R}$.

We say $(\mathcal{L}, \mathcal{R})$ factors $\mathcal{M}$ *functorially* if the assignments $f \mapsto i_f$ and $f \mapsto p_f$ give functors $i, p \colon \mathsf{Arr}(\mathcal{M}) \to \mathsf{Arr}(\mathcal{M})$ satisfying $\mathsf{src} \circ i = \mathsf{src}$, $\mathsf{targ} \circ p = \mathsf{targ}$, and $\mathsf{targ} \circ i = \mathsf{src} \circ p$, where src and targ denote the source and target functors $\mathsf{Arr}(\mathcal{M}) \to \mathcal{M}$ respectively.

Equivalently, a functorial factorization consists of a functor $Z \colon \mathsf{Arr}(\mathcal{M}) \to \mathcal{M}$ together with natural transformations $i \colon \mathsf{src} \Rightarrow Z$ and $p \colon Z \Rightarrow \mathsf{targ}$ such that the composite $\mathsf{src} \overset{i}{\Rightarrow} Z \overset{p}{\Rightarrow} \mathsf{targ}$ sends f to f.

3.4.4. **Weak factorization systems.** A *weak factorization system* (WFS) of $\mathcal{M}$ is an ordered pair $(\mathcal{L}, \mathcal{R})$ of collections of morphisms of $\mathcal{M}$ that factors $\mathcal{M}$ and satisfies $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$.

Lemma 3.4.4.1 ([MP12, Lemma 14.1.12])**.** *For a factorization $f = q \circ j \colon A \to B$ through an object Y in $\mathcal{M}$, if f has the LLP with respect to q, then f is a retract of j in* $\mathsf{Arr}(\mathcal{M})$*. Dually, if f has the RLP with respect to j, then f is a retract of q.*

Proposition 3.4.4.2 ([MP12, Proposition 14.1.13])**.** *Let $(\mathcal{L}, \mathcal{R})$ factor $\mathcal{M}$. Then $(\mathcal{L}, \mathcal{R})$ is a WFS if and only if $\mathcal{L} \boxslash \mathcal{R}$ and both $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts in* $\mathsf{Arr}(\mathcal{M})$*.*

A *strong factorization system* $(\mathcal{E}, \mathcal{M})$ of a category $\mathcal{C}$ is a weak factorization system of $\mathcal{C}$ such that all relevant lifts λ as in (3.4.2.1) are required to be unique. For example, $\mathcal{E} = \{$epimorphisms of $\mathcal{C}\}$ and $\mathcal{M} = \{$monomorphisms of $\mathcal{C}\}$. In fact, reversing the order, in many categories $(\mathcal{M}, \mathcal{E})$ is a weak but not a strong factorization system. We thus think of cofibrations as analogous to monomorphisms and fibrations as analogous to epimorphisms, and this will motivate Notation 3.4.5.1.

3.4.5. **Model structures.** A (Quillen) *model structure* on $\mathcal{M}$ is a triple $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ of collections of morphisms of $\mathcal{M}$, called *weak equivalences*, *cofibrations*, and *fibrations* respectively, satisfying the following axioms.

(M1) $\mathcal{W}$ has the 2-out-of-3 property.
(M2) $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is a weak factorization system.
(M3) $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is a weak factorization system.
(M4) The weak factorizations from (M2) and (M3) are functorial.

Maps in $\mathcal{F} \cap \mathcal{W}$ are called *trivial* (or *acyclic*) *fibrations* and maps in $\mathcal{C} \cap \mathcal{W}$ are called *trivial* (or *acyclic*) *cofibrations*. We call $\mathcal{M}$ a *model category* if it is equipped with a model structure.

Notation 3.4.5.1. We denote maps in $\mathcal{W}$, in $\mathcal{C}$, and in $\mathcal{F}$ by arrows of the form $\xrightarrow{\sim}$, $\hookrightarrow$, and $\twoheadrightarrow$ respectively.

We will see in Lemma 3.4.7.1 that this notation for weak equivalences in model categories agrees with the notation for categories for categories with weak equivalences from before.

Note that the cofibrations and fibrations of a model category $\mathcal{M}$ are the fibrations and cofibrations respectively of a model structure on the opposite category $\mathcal{M}^{\mathrm{op}}$. Thus our results about model categories come in dual pairs, so we only need to prove one.

We henceforth assume $\mathcal{M}$ has a model structure $(\mathcal{W}, \mathcal{C}, \mathcal{F})$. Then every morphism in $\mathcal{M}$ functorially factors both as the composite of a cofibration followed by an acyclic fibration and as an acyclic cofibration followed by a fibration. Moreover, there always exists a lift in the following commutative square whenever i is a cofibration, p is a fibration, and either i or p is acyclic.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\
X \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

That is,

$$\mathcal{C} = {}^{\boxslash}(\mathcal{F} \cap \mathcal{W}), \quad \mathcal{F} \cap \mathcal{W} = \mathcal{C}^{\boxslash}, \quad \mathcal{F} = (\mathcal{C} \cap \mathcal{W})^{\boxslash}, \quad \text{and} \quad \mathcal{C} \cap \mathcal{W} = {}^{\boxslash}\mathcal{F}. \tag{3.4.5.2}$$

Thus, to specify a model structure on a category with weak equivalences $(\mathcal{C}, \mathcal{W})$, we only need to specify either the cofibrations or the fibrations, but not both. Moreover, by Proposition 3.4.4.2, the equalities (3.4.5.2) are equivalent to the assertion that $\mathcal{C}$, $\mathcal{F}$, $\mathcal{C} \cap \mathcal{W}$, and $\mathcal{F} \cap \mathcal{W}$ are closed under retracts and satisfy $\mathcal{C} \boxslash (\mathcal{F} \cap \mathcal{W})$ and $\mathcal{F} \boxslash (\mathcal{C} \cap \mathcal{W})$. We thus have the following result.

Corollary 3.4.5.3. *A category $\mathcal{M}$ is a model category if and only if there are collections*

$\mathcal{W}$, $\mathcal{C}$, and $\mathcal{F}$ of morphisms in $\mathcal{M}$ that are closed under retracts in $\mathsf{Arr}(\mathcal{M})$ *and* $\mathcal{C} \boxslash (\mathcal{F} \cap \mathcal{W})$ *and* $\mathcal{F} \boxslash (\mathcal{C} \cap \mathcal{W})$.

3.4.6. **Examples of model categories.** TODO: model structure on Cat

3.4.7. **Model categories are also categories with weak equivalences.**

Lemma 3.4.7.1 ([MP12, Lemma 14.2.5])**.** *The class $\mathcal{W}$ as well as the classes $\mathcal{C}$, $\mathcal{C} \cap \mathcal{W}$, $\mathcal{F}$, and $\mathcal{F} \cap \mathcal{W}$ in a model structure $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ on $\mathcal{M}$ are subcategories that contain all isomorphisms and are closed under retracts. Therefore $\mathcal{W}$ is a subcategory of weak equivalences.*

3.4.8. **Fibrant and cofibrant objects and replacements.** An object X in a model category $\mathcal{M}$ is *cofibrant* if the unique map $\varnothing \to X$ is a cofibration. An acyclic fibration $q\colon QX \overset{\sim}{\twoheadrightarrow} X$ in which QX is cofibrant is called a *cofibrant approximation* or *cofibrant replacement* of X. Note that “replacement” is often reserved for functorial constructions while “approximation” is usually used when thinking about a single object. Here QX comes from the factorization $\varnothing \hookrightarrow QX \overset{\sim}{\twoheadrightarrow} X$. Next we show that all cofibrant replacements of X are weakly equivalent. Given two cofibrant replacements $q\colon QX \to X$ and $q'\colon Q'X \to X$, by considering the following diagram we get a map $\xi\colon QX \to Q'X$ such that $q' \circ \xi = q$.

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & Q'X \arrow[d, "q'"] \\
QX \arrow[r, "q"'] \arrow[ur, dashed, "\xi"] & X
\end{tikzcd}$$

Dually, $X \in \mathcal{M}$ is *fibrant* if the unique map $X \to *$ is a fibration. An acyclic cofibration $r\colon X \overset{\sim}{\hookrightarrow} RX$ in which RX is fibrant is called a *fibrant approximation* or *fibrant replacement* of X. To obtain RX, factor the fibration $X \twoheadrightarrow *$. Here RX comes from the factorization $X \overset{\sim}{\hookrightarrow} RX \twoheadrightarrow *$. A similar argument to the previous one shows that all fibrant replacements of an object are weakly equivalent.

3.4.9. **Construction of a “bifibrant” replacement.** Often we want to replace $X \in \mathcal{M}$ with an object that is both fibrant and cofibrant. The following plays an important role in the construction of the homotopy category $\mathsf{Ho}(\mathcal{M})$.

We will call $X \in \mathcal{M}$ *bifibrant* if it is both fibrant and cofibrant, though this terminology is nonstandard. Often one wants to replace X with a bifibrant object. There are two ways to do this from the above two constructions, which end up being weakly equivalent.

The first way is to take the factorization $\varnothing \hookrightarrow QX \overset{\sim}{\twoheadrightarrow} X$ and factor the second map $QX \overset{\sim}{\twoheadrightarrow} X$ as $QX \overset{\sim}{\hookrightarrow} RQX \twoheadrightarrow X$, so that we get the weak equivalence $X \sim RQX$ via the zig-zag of weak equivalences $X \overset{\sim}{\twoheadleftarrow} QX \overset{\sim}{\hookrightarrow} RXQ$.

The second way is to take the factorization $X \overset{\sim}{\hookrightarrow} RX \twoheadrightarrow *$ and factor the first map $X \overset{\sim}{\hookrightarrow} RX$ as $X \hookrightarrow QRX \overset{\sim}{\twoheadrightarrow} RX$ so that we get the weak equivalence $X \sim QXR$ via the zig-zag of weak equivalences $X \overset{\sim}{\hookrightarrow} RX \overset{\sim}{\twoheadleftarrow} QRX$. Thus $QXR \sim X \sim RXQ$, which shows QXR and RXQ are weakly equivalent in $\mathcal{M}$.

3.4.10. **Cylinder and path objects.** Here we follow [Str11, §10.2]. A *cylinder object* for an object $X \in \mathcal{M}$ is a factorization of the comparison morphism $\nabla\colon X \amalg X \to X$ (often called the *fold map*) of the form $X \amalg X \overset{i}{\hookrightarrow} \mathrm{Cyl}(X) \overset{\sim}{\to} X$. Dually, a *path object* (or *cocyclinder object*) for an object $Y \in \mathcal{M}$ is a factorization of the diagonal morphism $\Delta\colon Y \to Y \times Y$ of the form $Y \overset{j}{\hookrightarrow} \mathrm{Path}(Y) \overset{\sim}{\to} Y \times Y$.

Every object $X \in \mathcal{M}$ admits a cylinder object since we can factor ∇ as $X \amalg X \hookrightarrow QX \overset{\sim}{\twoheadrightarrow} X$, and every object $Y \in \mathcal{M}$ admits a path object because we can factor Δ as $Y \hookrightarrow Q(Y \times Y) \overset{\sim}{\twoheadrightarrow} Y \times Y$.

3.4.11. **Left and right homotopy of morphisms.** Let $f, g \in \mathcal{M}(X \to Y)$. We say f and g are *left homotopic* if there is a cylinder object $\mathrm{Cyl}(X) \in \mathcal{M}$ and a map $H \in \mathcal{M}(\mathrm{Cyl}(X) \to Y)$ such that $f \amalg g \in \mathcal{M}(X \amalg X \to Y)$ factors as $X \amalg X \hookrightarrow \mathrm{Cyl}(X) \xrightarrow{H} Y$.

We say f and g are *right homotopic* if there is a path object $\mathrm{Path}(Y) \in \mathcal{M}$ and a map $K \in \mathcal{M}(X \to \mathrm{Path}(Y))$ such that $f \times g \in \mathcal{M}(X \to Y \times Y)$ factors as $X \xrightarrow{K} \mathrm{Path}(Y) \overset{\sim}{\to} Y \times Y$.

We call f and g *homotopic* if they are both left and right homotopic.

In a general model category, left and right homotopy can be distinct relations, and neither is necessarily an equivalence relation. However, if the domains or targets are well-behaved (fibrant for targets, cofibrant for domains), then the two notions do coincide and are equivalence relations.

Proposition 3.4.11.1 ([Str11, Project 10.10, adapted])**.**

(i) *If X is cofibrant and $f, g \in \mathcal{M}(X \to Y)$ are left homotopic, then they are homotopic and left homotopy is an equivalence relation on $\mathcal{M}(X \to Y)$.*

(ii) *If Y is fibrant and $f, g \in \mathcal{M}(X \to Y)$ are right homotopic, then they are homotopic and right homotopy is an equivalence relation on $\mathcal{M}(X \to Y)$.*

It follows that if X is cofibrant and Y is fibrant, then homotopy is an equivalence relation on $\mathcal{M}(X \to Y)$. We write $f \simeq g$ to mean there is a homotopy (meaning either a left homotopy $H \in \mathcal{M}(\mathrm{Cyl}(X) \to Y)$ or a right homotopy $K \in \mathcal{M}(X \to \mathrm{Path}(Y))$) from f to g. We write $[X, Y]$ for the set $\mathcal{M}(X \to Y)/\simeq$ of equivalence classes of morphisms under homotopy (called *homotopy classes*).

Given this, it makes sense to focus on bifibrant objects in $\mathcal{M}$.

3.4.12. **Homotopy equivalence of bifibrant objects.** Bifibrant objects $X, Y \in \mathcal{M}$ are *homotopy equivalent* if there are morphisms $f \in \mathcal{M}(X \to Y)$ and $g \in \mathcal{M}(Y \to X)$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

Proposition 3.4.12.1 ([Str11, Problem 10.11])**.** *Bifibrant objects in $\mathcal{M}$ are weakly equivalent if and only if they are homotopy equivalent.*

3.4.13. **Homotopy-invariant functors.** For any category $\mathcal{C}$, a functor $F\colon \mathcal{M} \to \mathcal{C}$ is called *homotopy-invariant* if whenever f is a weak equivalence in $\mathcal{M}$, $F(f)$ is an isomorphism in $\mathcal{C}$.

Note that if $f, g \in \mathcal{M}(X \to Y)$ are left homotopic or right homotopic and $F\colon \mathcal{M} \to \mathcal{C}$ is homotopy-invariant, then $F(f) = F(g)$ in $\mathcal{C}$. One shows this by using the universal example $f = \iota_0\colon X \to \mathrm{Cyl}(X)$ and $g = \iota_1\colon X \to \mathrm{Cyl}(X)$; see [Str11, Problem 10.12].

3.4.14. **The homotopy category of a model category.** Let $\mathcal{M}_{cf}$ denote the full subcategory of $\mathcal{M}$ on the bifibrant objects of $\mathcal{M}$ and let $\mathsf{Ho}(\mathcal{M}_{cf})$ denote the category whose objects are the bifibrant objects of $\mathcal{M}$ and whose morphisms are homotopy classes of morphisms in $\mathcal{M}$, i.e., $\mathsf{Ho}(\mathcal{M}_{cf})(X \to Y) = [X, Y]$. We know from before that there is a functor $RQ\colon \mathcal{M} \to \mathsf{Ho}(\mathcal{M}_{cf})$ sending $X \in \mathcal{M}$ to a weakly equivalent bifibrant object $RQ(X) \in \mathcal{M}_{cf}$.

The *homotopy category* of the model category $\mathcal{M}$ is the category $\mathsf{Ho}(\mathcal{M})$ with the same objects as $\mathcal{M}$ and with morphisms $\mathsf{Ho}(\mathcal{M})(X \to Y) := [RQ(X), RQ(Y)]$ for all $X, Y \in \mathcal{M}$. As is suggested by our notation, there is a functor $\mathsf{Ho}\colon \mathcal{M} \to \mathsf{Ho}(\mathcal{M})$ given by $X \mapsto RQ(X)$ and $f \mapsto [RQ(f)]$.

Proposition 3.4.14.1 ([Str11, Theorem 10.14])**.** *The functor* $\mathsf{Ho}\colon \mathcal{M} \to \mathsf{Ho}(\mathcal{M})$ *is homotopy-invariant, and any other homotopy-invariant functor* $F\colon \mathcal{M} \to \mathcal{C}$ *factors uniquely through* $\mathcal{M} \overset{\mathsf{Ho}}{\to} \mathsf{Ho}(\mathcal{M})$.

3.4.15. **Derived functors.** Suppose we have a non-homotopy-invariant functor $F\colon \mathcal{M} \to \mathcal{C}$ into some category $\mathcal{C}$. By Proposition 3.4.14.1, this is equivalent to there being no functor $\Phi\colon \mathsf{Ho}(\mathcal{M}) \to \mathcal{C}$ that makes the following diagram commute.

$$\begin{tikzcd}
\mathcal{M} \arrow[rr, "F"] \arrow[dr, "\mathsf{Ho}"'] & & \mathcal{C} \\
& \mathsf{Ho}(\mathcal{M}) \arrow[ur, dashed, "\Phi"'] &
\end{tikzcd}$$

It would then be natural to ask whether we can find a “best homotopy-invariant approximation” of F. To do this, we relax the requirement that the above diagram of functors strictly commutes, so that we are now considering functors $\Phi\colon \mathsf{Ho}(\mathcal{M}) \to \mathcal{C}$ for which there is a natural transformation $\Phi \circ \mathsf{Ho} \Rightarrow F$. Now we ask the same question as before but in higher setting, i.e., we ask if there is a functor Φ and a natural transformation $\delta\colon \Phi \circ \mathsf{Ho} \Rightarrow F$ such that any other natural transformation $\Psi \circ \mathsf{Ho} \Rightarrow F$ factors uniquely through δ. As such a Φ satisfies this universal property, it is unique up to a unique natural isomorphism. If such a functor Φ exists, we call it the *left derived functor* of F and we denote it $\mathrm{Der}_L(F)\colon \mathsf{Ho}(\mathcal{M}) \to \mathcal{C}$.

Theorem 3.4.15.1 ([Str11, Project 10.16])**.** *Show that if $F(f)$ is an isomorphism for all weak equivalences $f \in \mathcal{M}(A \to B)$ and A and B are cofibrant, then $\mathrm{Der}_L(F)$ exists.*

Proposition 3.4.15.2 ([Str11, Problem 10.17])**.** *Show that if X is cofibrant, then $\mathrm{Der}_L(F)(X) \to F(X)$ is an equivalence.*

We define a *right derived functor* $\mathrm{Der}_R(F)$ of a functor $F\colon \mathcal{M} \to \mathcal{C}$ similarly.

3.4.16. **Total derived functors.** For a functor $F\colon \mathcal{M} \to \mathcal{N}$ between model categories $\mathcal{M}$ and $\mathcal{N}$, the *total left derived functor* $\mathbf{L}(F)\colon \mathsf{Ho}(\mathcal{M}) \to \mathsf{Ho}(\mathcal{N})$, if it exists, is the left derived functor of the composite $\mathcal{M} \xrightarrow{F} \mathcal{N} \xrightarrow{\mathsf{Ho}} \mathsf{Ho}(\mathcal{N})$.

We define a *total right derived functor* $\mathbf{R}(F)$ of a functor $F\colon \mathcal{M} \to \mathcal{N}$ similarly.

3.4.17. **(Quillen) equivalence of model categories.** We now define the correct notion of equivalence for model categories.

Theorem 3.4.17.1 ([Str11, Theorem 10.18 and Problem 10.20, adapted])**.** *Suppose $\mathcal{M}$ and $\mathcal{N}$ are model categories and $F\colon \mathcal{M} \to \mathcal{N}$ is left adjoint to $G\colon \mathcal{N} \to \mathcal{M}$.*

(i) *If F preserves cofibrations and G preserves fibrations, then the total derived functors $\mathbf{L}(F)\colon \mathsf{Ho}(\mathcal{M}) \to \mathsf{Ho}(\mathcal{N})$ and $\mathbf{R}(G)\colon \mathsf{Ho}(\mathcal{N}) \to \mathsf{Ho}(\mathcal{M})$ exist and $\mathbf{L}(F)$ is left adjoint to $\mathbf{R}(G)$.*

(ii) *F preserves both cofibrations and acyclic cofibrations if and only if G preserves both fibrations and acyclic cofibrations.*

We call an adjoint pair of functors $(F, G)\colon \mathcal{M} \to \mathcal{N}$ between model categories satisfying the condition of Theorem 3.4.17.1(ii) a *Quillen adjunction*.

Proposition 3.4.17.2 ([Str11, Problem 10.21, adapted])**.** *If (F, G) is a Quillen adjunction, then both F and G preserve weak equivalences.*

If a Quillen adjunction $(F, G)\colon \mathcal{M} \to \mathcal{N}$ has the property that for any $X \in \mathcal{M}$ and $Y \in \mathcal{N}$, a morphism $F(X) \to Y$ is a weak equivalence in $\mathcal{N}$ if and only if the corresponding map $X \to G(Y)$ is a weak equivalence in $\mathcal{M}$, then (F, G) is called a *Quillen equivalence*.

The following result shows that Quillen equivalence is the "correct" notion of equivalence between model categories.

Theorem 3.4.17.3 ([Str11, Project 10.22])**.** *If $(F, G)\colon \mathcal{M} \to \mathcal{N}$ is a Quillen equivalence, then the total derived functors $\mathbf{L}(F)\colon \mathsf{Ho}(\mathcal{M}) \to \mathsf{Ho}(\mathcal{N})$ and $\mathbf{R}(G)\colon \mathsf{Ho}(\mathcal{N}) \to \mathsf{Ho}(\mathcal{M})$ gives an equivalence of categories $\mathsf{Ho}(\mathcal{M}) \cong \mathsf{Ho}(\mathcal{N})$.*

Thus, if there is a Quillen equivalence between model categories $\mathcal{M}$ and $\mathcal{N}$, then any homotopy-theoretical question will have the same answer in both $\mathcal{M}$ and $\mathcal{N}$.

3.4.18. **Application: Homotopy colimits.** Fix a model category $\mathcal{M}$. If $\mathcal{J}$ is a simple category, then $\mathcal{M}^{\mathcal{J}}$ can be given the structure of a model category whose weak equivalences and fibrations are the pointwise weak homotopy equivalences and the pointwise fibrations, respectively. By Equation (3.4.5.2), this completely determines the cofibrations.

Unsurprisingly, the cofibrant objects in $\mathcal{M}^{\mathcal{J}}$ are the cofibrant diagrams in the usual sense, and by reasoning similar to our arguments in the case $\mathcal{M} = \mathcal{T}$ we can see that $F\colon \mathcal{J} \to \mathcal{M}$ is cofibrant if and only if each map $\operatorname{colim} F|_{\mathcal{J}_{<j}} \to F(j)$ is a cofibration in $\mathcal{M}$.

We now have the following problem, which is precisely the problem we had when defining the homotopy colimit when $\mathcal{M} = \mathcal{T}$.

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{J}} \arrow[r, "\operatorname{colim}"] \arrow[d, "\mathsf{Ho}"'] & \mathcal{M} \arrow[d, "\mathsf{Ho}"] \\
\mathsf{Ho}(\mathcal{M}^{\mathcal{J}}) \arrow[r, dashed] & \mathsf{Ho}(\mathcal{M})
\end{tikzcd}$$

Thus we define the *homotopy colimit* to be the total left derived functor $\mathbf{L}(\operatorname{colim})\colon \mathsf{Ho}(\mathcal{M}^{\mathcal{J}}) \to \mathsf{Ho}(\mathcal{M})$. By this definition, we get the homotopy-invariance of hocolim for free!

Theorem 3.4.18.1 ([Str11, Theorem 10.27])**.** *If $\mathcal{J}$ is a direct category, then* hocolim *exists and it is left adjoint to $\mathsf{Ho}(\Delta)\colon \mathcal{M} \to \mathcal{M}^{\mathcal{J}}$, where Δ is the constant diagram functor.*

Exercise 3.4.18.2 ([Str11, Problems 10.26–28, adapted])**.**

(i) Show that if the homotopy colimit exists, then pointwise weak equivalences $F \to G$ induce categorical equivalences of homotopy colimits.
(ii) Use Theorem 3.4.17.1(i) to prove Theorem 3.4.18.1.
(iii) Show that the rule $F \mapsto \operatorname{colim} \overline{F}$, where $\overline{F}$ is a cofibrant replacement for F, defines a total left derived functor for colim, thereby also proving Theorem 3.4.18.1.

3.5. CONSTRUCTIONS WITH HOMOTOPIES

3.5.1. **(Co-)H-spaces.** An *H-space*[2] (resp. *co-H-space*) is a unital magma object in $\mathsf{Ho}\,\mathcal{T}_*$ (resp. $(\mathsf{Ho}\,\mathcal{T}_*)^{\mathrm{op}}$). Unpacking this definition, we find $X \in \mathcal{T}_*$ is an H-space when equipped with a map $m\colon X \times X \to X$, called its *multiplication*, that makes the following diagram

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[2] "H" here is for "Hopf".

commute up to homotopy.

$$\begin{tikzcd}
X \vee X \arrow[d] \arrow[dr, "\nabla"] & \\
X \times X \arrow[r, "m"] & X
\end{tikzcd}$$

Here ∇ denotes the *fold map* given by $[(x,0)] \mapsto x$ on the first wedge summand and $[(x,1)] \mapsto x$ on the other. This definition says $*$ is a homotopy identity for m. One can thus view an H-space as a "homotopy unital magma".

A *homomorphism* of H-spaces (resp. co-H-spaces) is a pointed map $f\colon X \to Y$ that is compatible with the the multiplication maps up to homotopy, i.e., f makes the following diagram on the left (resp. right) commute in $\mathsf{Ho}\,\mathcal{T}_*$.

$$\begin{tikzcd}
X \times X \arrow[r, "f \times f"] \arrow[d, "m_X"] & Y \times Y \arrow[d, "m_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "c_X"] & Y \arrow[d, "c_Y"] \\
X \vee X \arrow[r, "f \vee f"] & Y \vee Y
\end{tikzcd}$$

Deformation retracts of H-spaces are H-spaces. If an H-space X is associative, then the functor $[-,X]_*$ takes values in the category of monoids and monoid homomorphisms. If m_X is homotopy commutative for an H-space X, then $[-,X]_*$ takes values in the category of commutative monoids. In particular, the fundamental group $\pi_1(X)$ of any H-space X is abelian.

3.5.2. **(Co-)H-groups.** A space $X \in \mathcal{T}_*$ is *H-group* (resp. *co-H-group*), or *grouplike* (resp. *co-H-group*), if it is a group object in $\mathsf{Ho}\,\mathcal{T}_*$ (resp. in $(\mathsf{Ho}\,\mathcal{T}_*)^{\mathrm{op}}$). A *topological group* is a strict H-group, i.e., an H-group space whose axioms are strict equalities instead of merely up to homotopy.

If $X \in \mathcal{T}_*$ and (Y,m) is an H-space, then $[X,Y]_*$ can be given an additive structure as follows. For $f,g\colon X \to Y$, define $f+g := m \circ (f \times g) \circ \Delta\colon X \to Y$, where $\Delta\colon X \to X \times X$ is the *diagonal map* given by $\Delta(x) := (x,x)$. Then $[f]+[g] := [f+g]$ is a well-defined binary operation on $[X,Y]_*$.

Proposition 3.5.2.1 ([Ark11, Proposition 2.2.6, adapted])**.** *If (Y,m) and (Y',m') are H-spaces and $h\colon (Y,m) \to (Y',m')$ is an H-map, then postcomposition $h_*\colon [X,Y]_* \to [X,Y']_*$ is a map of pointed sets with a binary operation '+' defined above.*

In particular, if Y and Y' are H-groups, then $h_*\colon [X,Y]_* \to [X,Y']_*$ is a group homomorphism.

Proof. For $[a],[b] \in [X,Y]$, we have $h \circ (a+b) = h \circ m \circ (a \times b) \circ \Delta \simeq m' \circ (h \times h) \circ (a \times b) \circ \Delta = h \circ a + h \circ b$, so h_* is a group homomorphism. □

Proposition 3.5.2.2. *For $Y \in \mathcal{T}_*$, Y is an H-group if and only if the functor $[-, Y]_* \colon \mathcal{T}_* \to \mathsf{Set}_*$ factors through Grp.*

Proposition 3.5.2.3 ([Ark11, Exercise 2.1, adapted])**.** *For an H-space (Y, m) such that the pair $(Y \times Y, Y \vee Y)$ has the HEP, there is a multiplication m' on Y such that $m' \simeq m$ and $m'(y, *) = y$ and $m'(*, y) = y$ for all $y \in Y$.*

See [Ark11, §2.2].

Exercise 3.5.2.4 ([Ark11, Exercise 2.6, adapted])**.** Prove that a space X admits a comultiplication if and only if the diagonal map $\Delta \colon X \to X \times X$ can be factored up to homotopy through $X \vee X$.

Exercise 3.5.2.5 ([Ark11, Exercise 2.9, adapted])**.** Let (X, c) and (X', c') be co-H-spaces and let $g \colon X' \to X$ be a map. Prove that g is a co-H-map if and only if for every space Y, and every $\alpha, \beta \in [X, Y]$, we have $(\alpha + \beta)[g] = \alpha_g + \beta_g$.

3.5.3. **(Co)actions and their relation to (co)fibrations.** Here we follow [Ark11, §§4.3–4], which discusses (co)actions and how they relate to fibration and cofibration sequences. Given an H-space (X, m) and a pointed space $A \in \mathcal{T}_*$, a *right action* of (X, m) on A is a right X-action on A in $\mathsf{Ho}\,\mathcal{T}_*$, i.e., a map $\phi \colon X \times A \to A$ such that (i) $\phi \circ j_1 \simeq \mathrm{id}_X \colon X \to X$ for the inclusion $j_1 \colon X \hookrightarrow X \times A$, and (ii) $\phi(\phi \times \mathrm{id}_A) \simeq \phi \circ (\mathrm{id}_X \times m) \colon X \times A \times A \to X$. By dualizing this definition we obtain the notion of a right *coaction* of a co-H-space on a pointed space.

Proposition 3.5.3.1 ([Ark11, Proposition 4.3.6, adapted])**.** *Given a map $f \colon X \to Y$, construct and prove that there is a coaction $\phi \colon Cf \to Cf \vee \Sigma X$.*

Proposition 3.5.3.2 ([Ark11, Proposition 2.2.3, adapted])**.** *Fix $Y \in \mathcal{T}_*$.*

- (i) *Y is an H-space if and only if $[-, Y] \colon \mathcal{T}_* \to \mathsf{Set}_*$ factors through the category $\mathsf{Mag_u}$ of unital magmas such that the set $[X, Y]$ is a pointed set having a binary operation for which the homotopy class of the constant map is a 2-sided identity and that $f^* \colon [X', Y] \to [X, Y]$ is a unital magma homomorphism for every map $f \colon X \to X'$.*
- (ii) *Y is an H-group if and only if $[-, Y]$ factors through the category Grp of groups in a similar way to (i).*

Proof. We leave (i) as an exercise, along with the details for (ii), but we do note that if $[-, Y]$ factors through Grp, then Y is an H-group with homotopy multiplication $m \colon Y \times Y \to Y$ and homotopy inverse $i \colon Y \to Y$ given by $[m] := [p_1] + [p_2]$ and $+$ is the multiplication on the image of Y under the functor $\mathcal{T}_* \to \mathsf{Grp}$, and where where p_1 and p_2 are the projections $Y \times Y \to Y$ and $[i] = -[\mathrm{id}_Y]$ □

3.6. Diagram homotopies

3.6.1. **Warmup 1: Homotopy equivalence of maps.** The following category can be defined when $\mathcal{T}$ is replaced by any category $\mathcal{C}$. Let $\mathsf{Arr}(\mathcal{T})$ be the category whose objects are maps in $\mathcal{T}$ and whose morphisms $\mathsf{Arr}(\mathcal{T})(f \to g)$ are pairs $\alpha = (\alpha^d, \alpha_t)$ (d for domain, t for target) that make the following diagram commute in $\mathcal{T}$.

$$\begin{tikzcd}
X \arrow[r, "\alpha^d"] \arrow[d, "f"] & Z \arrow[d, "g"] \\
Y \arrow[r, "\alpha_t"] & W
\end{tikzcd}$$

For an object $f\colon X \to Y$ in $\mathsf{Arr}(\mathcal{T})$, the cylinder $\mathrm{Cyl}(f)$ is in $\mathcal{T}$ and comes equipped with inclusion morphisms $(\iota_0, \iota_0), (\iota_1, \iota_1) \in \mathsf{Arr}(\mathcal{T})(f \to \mathrm{Cyl}(f))$ given in the following diagrams.

$$\begin{tikzcd}
X \arrow[r, hook, "\iota_0"] \arrow[d, "f"] & \mathrm{Cyl}(X) \arrow[d, "\mathrm{Cyl}(f)"] \\
Y \arrow[r, hook, "\iota_0"] & \mathrm{Cyl}(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, hook, "\iota_1"] \arrow[d, "f"] & \mathrm{Cyl}(X) \arrow[d, "\mathrm{Cyl}(f)"] \\
Y \arrow[r, hook, "\iota_1"] & \mathrm{Cyl}(Y)
\end{tikzcd}$$

Morphisms $\alpha, \beta \in \mathsf{Arr}(\mathcal{T})(f \to g)$ are *homotopic* in $\mathsf{Arr}(\mathcal{T})$ if there is a *homotopy* $H\colon \alpha \simeq \beta$, which we define to be a morphism $H \in \mathsf{Arr}(\mathcal{T})(\mathrm{Cyl}(f) \to g)$, making the following diagram commute in $\mathsf{Arr}(\mathcal{T})$.

$$\begin{tikzcd}
f \arrow[r, hook, "\iota_0"] \arrow[dr, swap, "\alpha"] & \mathrm{Cyl}(f) \arrow[d, "H"] & f \arrow[l, hook', swap, "\iota_1"] \arrow[dl, "\beta"] \\
 & g &
\end{tikzcd}$$

One unpacks this definition to obtain that a homotopy $H\colon \alpha \simeq \beta$ in $\mathsf{Arr}(\mathcal{T})$ is a pair $(H^d\colon \alpha^d \simeq \beta^d, H_t\colon \alpha_t \simeq \beta_t)$ of homotopies in $\mathcal{T}$ that are compatible in the sense that the following diagram commutes in $\mathcal{T}$.

$$\begin{tikzcd}
\mathrm{Cyl}(X) \arrow[r, "H^d"] \arrow[d, swap, "\mathrm{Cyl}(f)"] & Z \arrow[d, "g"] \\
\mathrm{Cyl}(Y) \arrow[r, "H_t"] & W
\end{tikzcd}$$

Here we call H_t a *homotopy under* H^d and say $H = (H^d, H_t)$ is a pair of *coherent homotopies*.

A morphism $\alpha \in \mathsf{Arr}(\mathcal{T})(f \to g)$ is a *homotopy equivalence* of maps in $\mathsf{Arr}(\mathcal{T})$ if there is a morphism $\beta \in \mathsf{Arr}(\mathcal{T})(g \to f)$ and homotopies $H\colon \alpha \circ \beta \simeq \mathrm{id}_g$ and $K\colon \beta \circ \alpha \simeq \mathrm{id}_f$ in $\mathsf{Arr}(\mathcal{T})$.

3.6.2. **Warmup 2: Pointwise homotopy equivalence of maps.** For many applications, homotopy equivalence in $\mathsf{Arr}(\mathcal{T})$ is too strong of a condition. A useful weakening is the following.

Definition 3.6.2.1. A morphism $\alpha \in \mathsf{Arr}(\mathcal{T})(f \to g)$ is a *pointwise homotopy equivalence*[3] of maps in $\mathcal{T}$ if α^d and α_t are homotopy equivalences in $\mathcal{T}$.

We say maps in $\mathcal{T}$ are *pointwise homotopy equivalent* in $\mathcal{T}$ if there is a chain of pointwise homotopy equivalences $f = f_0 \leftarrow f_1 \to f_2 \leftarrow \cdots \to f_n = g$ in $\mathsf{Arr}(\mathcal{T})$. This definition seems strange, but really weak equivalence between spaces is just defined to be the equivalence relation *generated* by weak equivalences. We define it this way because the existence of a weak homotopy equivalence from one space to another is not an equivalence relation since it is not symmetric; see here for a counterexample of symmetry.

A *pointwise equivalence* of maps in $\mathsf{Ho}\,\mathcal{T}$ is a morphism (object) $\alpha = (\alpha^d, \alpha_t)$ in $\mathsf{Ho}(\mathsf{Arr}(\mathcal{T}))$ (that is, α is only required to make a certain diagram homotopy commutative) such that α^d and α_t are homotopy equivalences in $\mathcal{T}$. Maps that are equivalent in $\mathsf{Ho}\,\mathcal{T}_*$ produce "the same" map on sets of homotopy classes; see [Str11, Problem 4.76].

3.6.3. **Homotopy equivalence of diagrams.** We now generalize (pointwise) homotopy equivalences of maps to general (small) diagrams. Just like $\mathsf{Arr}(\mathcal{T})$, following category can be defined when $\mathcal{T}$ is replaced by any category $\mathcal{C}$. Fix a small category $\mathcal{J}$ and let $\mathcal{T}^{\mathcal{J}}$ denote the category of $\mathcal{J}$-shaped diagrams in $\mathcal{T}$ (i.e., functors $\mathcal{J} \to \mathcal{T}$) whose morphisms $\mathcal{T}^{\mathcal{J}}(F \to G)$, called *diagram morphisms*, are natural transformations $F \Rightarrow G$. For example, when $\mathcal{J}$ is the category with exactly two objects and a single non-identity morphism between them, i.e., we recover $\mathsf{Arr}(\mathcal{T})$ from above.

For a diagram $F \in \mathcal{T}^{\mathcal{J}}$, the *cylinder* $\mathrm{Cyl}(F)$ is the diagram $\mathrm{Cyl} \circ F$, where Cyl is the usual cylinder functor on $\mathcal{T}$. For example, when $\mathcal{J}$ is the aforementioned 2-object diagram and $F \in \mathcal{T}_\circ^{\mathcal{J}}$ is the diagram $X \xrightarrow{f} Y$ in $\mathcal{T}_\circ$, $\mathrm{Cyl}(F)$ takes the form $X \times I \xrightarrow{f \times \mathrm{id}_I} Y \times I$. As in the case of $\mathsf{Arr}(\mathcal{T})$, each component morphism of $\mathcal{J}$ comes equipped with inclusion morphisms ι_0 and ι_1.

Diagram morphisms $\alpha, \beta \in \mathcal{T}^{\mathcal{J}}(F \to G)$ are *homotopic* in $\mathcal{T}^{\mathcal{J}}$ if there is a *homotopy* $H \colon \alpha \simeq \beta$, that is, a morphism $H \in \mathcal{T}^{\mathcal{J}}(\mathrm{Cyl}(F) \to G)$ such that $H \circ \iota_0 = \alpha$ and $H \circ \iota_1 = \beta$. Equivalently, H is a collection of homotopies $\{H_j \colon \alpha_j \to \beta_j\}_{j \in \mathcal{J}}$ in $\mathcal{T}$ such that the following diagram commutes for all $k \in \mathcal{J}(j \to j')$.

$$\begin{tikzcd}
\mathrm{Cyl}(F(j)) \arrow[r, "\mathrm{Cyl}(F(k))"] \arrow[d, "H_j"'] & \mathrm{Cyl}(F(j')) \arrow[d, "H_{j'}"] \\
G(j) \arrow[r, "G(k)"] & G(j')
\end{tikzcd} \tag{3.6.3.1}$$

Diagrams $F, G \in \mathcal{T}^{\mathcal{J}}$ are *homotopy equivalent* if there are diagram morphisms $\alpha \colon F \to G$, $\beta \colon G \to F$ and diagram homotopies $\beta \circ \alpha \simeq \mathrm{id}_F$, $\alpha \circ \beta \simeq \mathrm{id}_G$.

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[3] Really, we should call this "objectwise" or perhaps "componentwise" instead of "pointwise".

3.6.4. **Pointwise homotopy equivalence of diagrams.** As was the case for $\mathsf{Arr}(\mathcal{T})$, for many applications, diagram homotopy equivalence is too strong of a condition, so we usually work with the following weaker condition. A morphism $\alpha \in \mathcal{T}^{\mathcal{J}}(F \to G)$ is a *pointwise* (or *objectwise*) *homotopy equivalence* of diagrams in $\mathcal{T}$ if $\alpha_j \colon F(j) \to G(j)$ is a homotopy equivalence of objects in $\mathcal{T}$ for all $j \in \mathcal{J}$. Every diagram homotopy equivalence is a pointwise homotopy equivalence.

Warning 3.6.4.1. Although pointwise homotopy equivalences α of diagrams have component maps α_j each having an homotopy inverse in $\mathcal{T}$, this does *not* mean α itself has a homotopy inverse. As Strom puts it [Str11, p. 114, notation adapted], "If α is a pointwise homotopy equivalence, then each $\alpha(j)$ has a homotopy inverse in $\mathcal{T}$, but no claims are made about how those homotopy inverses are related to one another or about the homotopies that are implicit in the assertion that they are homotopy inverses". An example of this failure can be found by noting $S^n \not\simeq *$ and considering the pushout of the diagram $CX \hookleftarrow X \hookrightarrow CX$.

3.7. Homotopy colimits

Our main resource for Homotopy colimits will be [Str11, Chapter 6], although we will frequently diverge from this.

3.7.1. **Warmup: functoriality of colimits.** First we review the functoriality of the usual colimit functor. Fix a cocomplete category and an index category $\mathcal{J}$. Notice that taking colimits is functorial in the sense that colim is a functor $\mathcal{C}^{\mathcal{J}} \to \mathcal{C}$. Indeed, for a diagram morphism $\alpha \in \mathcal{T}^{\mathcal{J}}(F \to G)$, the following solid diagram commutes.

$$\begin{tikzcd}
F(j) \arrow[rr, "F(k)"] \arrow[dr] \arrow[dd, "\alpha_j"'] & & F(j') \arrow[dl] \arrow[dd, "\alpha_j"] \\
& \operatorname{colim} F \arrow[dd, dashed, "\exists!\alpha_*"] & \\
G(j) \arrow[rr, "G(k)"] \arrow[dr] & & G(j') \arrow[dl] \\
& \operatorname{colim} G &
\end{tikzcd}$$

Thus $\operatorname{colim} G$ is the vertex of a cone under the diagram F, so by the universal property of $\operatorname{colim} F$ there is a unique morphism $\alpha_* \colon \operatorname{colim} F \to \operatorname{colim} G$ for which the entire diagram commutes. And given a diagram morphism $\beta \in \mathcal{T}^{\mathcal{J}}(G \to H)$, by adding to the above diagram a square for G and H we see by uniqueness of the induced comparison map that $(\beta \circ \alpha)_* = \beta_* \alpha_*$, where $\beta \circ \alpha$ denotes the vertical composite, i.e., $(\alpha \circ \beta)_j := \beta_j \alpha_j \colon F(j) \to H(j)$. One similarly shows the identity natural transformation $\mathrm{id}_F \colon F \Rightarrow F$ induces the identity map on colimits.

3.7.2. **Motivation for homotopy colimits.** It is frequent in homotopy theory that we want to compare the colimits of two diagrams using a pointwise homotopy equivalence between them. Unfortunately, it is not true in general that pointwise homotopy equivalence induces a homotopy equivalence of the corresponding colimits. It is thus natural to ask whether there is a functor $\operatorname{hocolim}\colon \mathcal{T}^{\mathcal{J}} \to \mathcal{T}$ that sends diagrams to the "best homotopy-invariant approximation" to colim in the sense that the following properties hold.

(HC1) Pointwise homotopy equivalences $F \simeq G$ in $\mathcal{T}^{\mathcal{J}}$ induce homotopy equivalences $\operatorname{hocolim} F \simeq \operatorname{hocolim} G$ in $\mathcal{T}$.

(HC2) There is a natural comparison map $\xi_F\colon \operatorname{hocolim} F \to \operatorname{colim} F$.

(HC3) Any other functor satisfying (HC1) and (HC2) factors uniquely through hocolim.

In short, we want hocolim to be a certain right Kan extension (in fact, a right derived functor).

3.7.3. **Plan to define homotopy colimits of spaces.** Despite the general failure of pointwise homotopy equivalences to induce homotopy equivalences of colimits in $\mathcal{T}$, it remains worthwhile to identify the diagrams for which this property holds. We will discover such diagrams are precisely *cofibrant* diagrams, i.e., those cofibrant objects in the model category of diagrams in $\mathcal{T}$. We will then show that most diagrams F in practice turn out to be pointwise homotopy equivalent to one of these nice diagrams $\overline{F}$, i.e., admit *cofibrant replacements* $\overline{F} \to F$. We will then define a *homotopy colimit* of F to be any space $X \in \mathcal{T}$ that is homotopy equivalent to $\operatorname{colim} \overline{F}$ for some cofibrant replacement $\overline{F} \to F$, and we will write $X = \operatorname{hocolim} F$ to indicate this.

At this point, we will show $\operatorname{hocolim} F$ is defined up to homotopy equivalence in $\mathcal{T}$: [Str11, Problem 6.16(c)] (Exercise 3.9.3.1(c)) says there is a diagram homotopy equivalence $\overline{F} \simeq \widehat{F}$, which in turn gives a homotopy equivalence $\operatorname{colim} \overline{F} \simeq \operatorname{colim} \widehat{F}$ by [Str11, Proposition 6.5] (Proposition 3.9.1.2). We often speak of *the* homotopy colimit of F, but we really mean up to homotopy equivalence.

Note that hocolim as defined above satisfies (HC1) because [Str11, Problem 6.16(b)] (Exercise 3.9.3.1(b)) says that pointwise homotopy equivalences $F \to G$ give diagram homotopy equivalences $\overline{F} \to \overline{G}$, which by [Str11, Proposition 6.5] (Proposition 3.9.1.2) gives homotopy equivalences $\operatorname{colim} \overline{F} \to \operatorname{colim} \overline{G}$ in $\mathcal{T}$. Also hocolim satisfies (HC2), since a cofibrant replacement $\overline{F} \to F$ gives, by the functoriality of colim, a comparison map $\xi_F\colon \operatorname{colim} \overline{F} \to \operatorname{colim} F$.

3.7.4. **Plan to make the homotopy colimit functorial.** As it stands, the above construction is not functorial in the sense that a diagram morphism $\phi\colon F \to G$ need not induce a well-defined homotopy class of maps $\Phi\colon \operatorname{hocolim} F \to \operatorname{hocolim} G$; though we can something in [Str11, Problem 6.23] (Exercise 3.9.6.1) and [Str11, Problem 6.21] (Exercise 3.9.5.1).

The failure of functoriality of the above construction stems from the fact that we are free to choose *any* cofibrant replacement of a diagram; to ensure functoriality, we need a consistent way to to construct cofibrant replacements $\overline{F}$ from given diagrams F. As one might suspect, "consistent" here means that our choice of cofibrant replacements should be functorial.

To that end, suppose that for an index category $\mathcal{J}$ we have constructed a functor $\mathrm{Cof}\colon \mathcal{T}^{\mathcal{J}} \to \mathcal{T}^{\mathcal{J}}$ and a natural transformation $\xi\colon \mathrm{Cof} \Rightarrow \mathrm{id}_{\mathcal{T}^{\mathcal{J}}}$ such that for all diagrams $F \in \mathcal{T}^{\mathcal{J}}$, the map $\xi_F\colon \mathrm{Cof}(F) \to F$ in $\mathcal{T}$ is a cofibrant replacement. (We will construct such a functor in Construction 3.7.5.2.) Fortunately, [Str11, Theorem 6.45] says this can be done when $\mathcal{J}$ is a *simple*, i.e., is a poset category $(\mathcal{J}, \leq)$ whose objects cannot be upper bounds of nontrivial linearly ordered chains in $\mathcal{J}$ of infinite length; most everyday index categories are simple. We then define the *homotopy colimit functor* $\mathrm{hocolim}\colon \mathcal{T}^{\mathcal{J}} \to \mathcal{T}$ by $\mathrm{hocolim} := \mathrm{colim} \circ \mathrm{Cof}$. Since $\mathrm{Cof}(F) \to F$ is a cofibrant replacement, $\mathrm{hocolim}\, F$ is the same homotopy colimit as before, and thus satisfies (HC1) and (HC2). But now [Str11, Theorem 6.27] (Theorem 3.9.8.1) below asserts that hocolim satisfies (HC3) as well, so we have achieved our initial goal, at least for simple diagram categories.

3.7.5. **Plan to functorially construct cofibrant replacements.** Fix a simple diagram $\mathcal{J}$.[4] A diagram $F \in \mathcal{T}^{\mathcal{J}}$ is cofibrant if and only if for all diagram maps $\alpha\colon F \to Y$ and pointwise acyclic fibrations $\pi\colon X \to Y$, there is a solution to the following problem on the left in $\mathcal{T}$.

$$\begin{tikzcd}
 & X \arrow[d, "\pi"] \\
F \arrow[r, "\alpha"] \arrow[ur, dashed] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathrm{colim}\, F_{<j} \arrow[r] \arrow[d, "\xi_j"'] & E(j) \arrow[d, "\pi_j"] \\
F(j) \arrow[r] \arrow[ur, dashed] & B(j)
\end{tikzcd}$$

By [Str11, Problem 6.33], this is equivalent to the set of problems in $\mathcal{T}$ of the above form on the right for each $j \in \mathcal{J}_{n+1} \smallsetminus \mathcal{J}_n$, where $\xi_j\colon \mathrm{colim}\, F_{<j} \to F(j)$ is the natural comparison map. Such a map exists when ξ_j is a cofibration in $\mathcal{T}$ for each $j \in \mathcal{J}$. This gives the following.

Theorem 3.7.5.1 ([Str11, Theorem 6.36 and Problem 6.38])**.** *For a diagram $F\colon \mathcal{J} \to \mathcal{T}$ indexed by a simple category $\mathcal{J}$, if the comparison map $\xi_j\colon \mathrm{colim}\, F_{<j} \to F(j)$ is a cofibration in $\mathcal{T}$ (and $F(j)$ is well-pointed when $\mathcal{T} = \mathcal{T}_*$) for each $j \in \mathcal{J}$, then F is a cofibrant diagram. If $\mathcal{T} = \mathcal{T}_\circ$, then the converse is also true.*

This suggests we construct the cofibrant replacement diagram $\mathrm{Cof}(F)$ and the pointwise homotopy equivalence $\xi_F\colon \mathrm{Cof}(F) \to F$ in $\mathcal{T}^{\mathcal{J}}$ as follows.

Construction 3.7.5.2.

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[4] In the (unlikely) scenario one wants to compute the homotopy colimit of a non-simple diagram, they may consult [Str11, §6.8].

(i) Write $\mathcal{J}$ as a poset category $(\mathcal{J}, \leq)$ whose objects are not upper bounds of nontrivial linearly ordered chains in $\mathcal{J}$ of infinite length. For each $n \geq 0$, let $\mathcal{J}_n$ be the full subcategory of $\mathcal{J}$ on the objects $\{j \in \mathcal{J} \mid d(j) \leq n\} \subseteq \mathcal{J}$.

(ii) Do the following for each $j \in \mathcal{J}_1 \smallsetminus \mathcal{J}_0$.

- Let $\mathcal{J}_{<j}$ be the full subcategory of $\mathcal{J}$ on the objects $\{j' \in \mathcal{J} \mid j' \leq j \text{ and } j' \neq j\} \subseteq \mathcal{J}$.
- Set $\overline{F}(j) \coloneqq F(j)$ and consider the cofibrant replacement $\overline{F}|_{\mathcal{J}_0} \to F|_{\mathcal{J}_0}$ given by the identity transformation.
- Compute $\operatorname{colim} F|_{\mathcal{J}_{<j}}$ and let $\xi_j \colon \operatorname{colim} F_{<j} \to F(j)$ be the natural comparison map.
- Set $\overline{F}(j) \coloneqq M_{\xi_j}$ and let $\overline{\xi}_j \colon \operatorname{colim} F_{<j}$ be the map obtained by replacing ξ_j with a cofibration in the usual way.

(iii) Repeat the above step for each $j \in \mathcal{J}_2 \smallsetminus \mathcal{J}_1$, and then again for each $j \in \mathcal{J}_3 \smallsetminus \mathcal{J}_2$, and so on, to inductively obtain a cofibrant replacement $\xi_F \colon \operatorname{Cof}(F) \to F$ in $\mathcal{T}$.

(iv) Set $\operatorname{hocolim} F \coloneqq \operatorname{colim} \operatorname{Cof}(F)$.

It is then a straightforward check that $\operatorname{Cof}(F)$ and ξ gives a functor $\operatorname{Cof} \colon \mathcal{T}^{\mathcal{J}} \to \mathcal{T}^{\mathcal{J}}$ and a natural transformation $\xi \colon \operatorname{Cof} \Rightarrow \mathrm{id}_{\mathcal{T}^{\mathcal{J}}}$ such that for all diagrams $F \in \mathcal{T}^{\mathcal{J}}$, the map $\xi_F \colon \operatorname{Cof}(F) \to F$ in $\mathcal{T}$ is a cofibrant replacement.

3.7.6. **Example: Homotopy pushout diagrams.** We follow the above procedure to compute $\operatorname{hocolim}(Y \xleftarrow{f} X \xrightarrow{g} Z)$ in $\mathcal{T}_\circ$. For the next step, first note $\operatorname{colim} F_{<*} = \operatorname{colim}(X) = (X)$ and $\operatorname{colim} F_{<\circ} = \operatorname{colim}(X) = (X)$, so the comparison maps $\xi_* \colon X \to Y$ and $\xi_\circ \colon X \to Z$ are just the original maps f and g respectively. Thus we only need to replace f and g with cofibrations, so $\operatorname{Cof}(Y \xleftarrow{f} X \xrightarrow{g} Z) = M_f \hookleftarrow X \to M_g$. Thus $\operatorname{hocolim}(Y \xleftarrow{f} X \xrightarrow{g} Z) = \operatorname{colim}(M_f \hookleftarrow X \hookrightarrow M_g)$, which is sometimes called the *double mapping cylinder* of f and g for obvious geometric reasons.

3.7.7. **Easy cofibrant replacements for treelike diagrams.** As is unsurprising from the above example, the slogan "replace all maps with cofibrations" is often confused with the actual procedure "replace the diagram with a cofibrant one". Although the technique used to build a cofibrant replacement for a prepushout diagram is to replace all the maps with pointwise homotopy equivalent cofibrations, this does *not* work in general, even for simple diagrams! For example, the diagram $X \rightrightarrows X$ in $\mathcal{T}$ where both maps are the identity is not cofibrant.

However, if the category $\mathcal{J}$ is *treelike* in the sense that each object $j \in \mathcal{J}$ that is not a root has a unique predecessor—which we may thus formally denote by $j - 1$—then this process does work. The prepushout diagram is treelike, which is why this works for them.

Exercise 3.7.7.1 ([Str11, Exercise 6.42])**.** Let $F\colon \mathcal{J} \to \mathcal{T}$ and when $\mathcal{T} = \mathcal{T}_*$ assume that $F(j)$ is well-pointed for each $j \in \mathcal{J}$.

(a) Show that if $\mathcal{J}$ is tree-like, then $\operatorname{colim} F_{<j} = F(j-1)$.

(b) Show that F is cofibrant if and only if every map $F(j \to j')$ is a cofibration.

Proof. For (a), note $\mathcal{J}_{<j}$ is a linearly ordered chain of finite length with terminal object $F(j-1)$, and the colimit of such a chain is the final object $F(j-1)$. For (b), notice Theorem 3.7.5.1 and (a) imply F is cofibrant if $\xi_j\colon F(j-1) \to F(j)$ is a cofibration, but all morphisms in $\mathcal{J}$ take this form. □

3.8. EXAMPLE COMPUTATIONS

Exercise 3.8.0.1 ([Str11, Problem 6.48])**.**

(i) Introduce an equivalence relation $\sim$ on the space $(C \amalg (A \rtimes I) \amalg B)$ by setting $(a,0) \sim g(a)$ and $(a,1) \sim f(a)$. Show that the quotient space $M(f,g) = (C \amalg (A \rtimes I) \amalg B)/\sim$ is a homotopy pushout for the diagram $C \leftarrow A \to B$. Is $M(f,g)$ equal to the standard homotopy colimit? Is it homeomorphic to the standard homotopy colimit?

(ii) Show that the pushout of $C \leftarrow A \to M_f$ is a homotopy colimit for $C \leftarrow A \to B$.

The space $M(f,g)$ is often referred to as the *double mapping cylinder* of the maps f and g. The result of (ii) implies that, to construct a homotopy pushout, it suffices to convert just one of the two maps to a cofibration.

Exercise 3.8.0.2 ([Str11, Problem 6.54])**.** Show that if either A or B is well-pointed, then $A \vee B$ is a homotopy pushout for $B \leftarrow * \to A$. What is the homotopy pushout when neither space is cofibrant?

Exercise 3.8.0.3 ([Str11, Problem 6.55])**.**

(i) Show that ΣA is the homotopy pushout of the diagram $* \leftarrow A \to *$.

(ii) Show that the homotopy colimit of $* \leftarrow A \xrightarrow{f} B$ is the homotopy cofiber of f.

Solution. (i) $\operatorname{hocolim}(* \leftarrow A \to *) = \operatorname{colim}(M_* \hookleftarrow A \hookrightarrow M_*) =$ double mapping cone $= \Sigma X$.

(ii) $\operatorname{hocolim}(* \leftarrow X \xrightarrow{f} Y) = \operatorname{colim}(M_* \hookleftarrow X \hookrightarrow M_f) =$ double mapping cylinder $M(*, f) = Cf$. □

Exercise 3.8.0.4. Given a diagram of spaces $X \to Z \leftarrow Y$, determine and prove what the homotopy pullback of this diagram is.

Proof. Since this is “cotreelike’ (or “opposite treelike”), just replace the maps with fibrations and take the limit. That is, where $E_f = \{(x, \alpha) \mid x \in X, \alpha \in \mathrm{Path}(Z), \alpha(0) = f(x)\}$,

$$\begin{aligned}\mathrm{hocolim}(X \xrightarrow{f} Z \xleftarrow{g} Y) &= \lim(E_g \twoheadrightarrow Z \twoheadleftarrow E_f) = E_g \times_Z E_f \\ &= \{(x, y, \alpha, \beta) \mid \alpha(0) = f(x), \beta(0) = g(y), \alpha(1) = \beta(1)(= f(x) = g(y))\} \\ &= \{(x, y, \gamma) \mid \gamma(0) = f(x), \gamma(1) = g(y)\}\end{aligned}$$

where we set $\gamma \coloneqq \alpha * \overline{\beta}$ to get the last equality. □

Exercise 3.8.0.5 ([Str11, Problem 6.85])**.** Determine the homotopy pullbacks of the following diagrams.

(i) $* \to X \leftarrow *$.
(ii) $X \to * \leftarrow Y$.

Solution. Let $z, z' \in Z$.

(i) $\mathrm{holim}(* \xrightarrow{c_z} Z \xleftarrow{c_{z'}} *) = \lim(E_{c_z} \overset{\mathrm{ev}_1}{\twoheadrightarrow} Z \overset{\mathrm{ev}_1}{\twoheadleftarrow} E_{c_{z'}}) = \Pi_1(Z)(z \to z')$, where the last equality is by the previous problem.
(ii) $\mathrm{holim}(X \to * \leftarrow Y) = \{(x, y, \gamma) \in X \times Y \times \mathrm{Path}(*) \mid \gamma(0) = *, \gamma(1) = *\} \cong X \times Y$. □

Exercise 3.8.0.6. Show that the homotopy fiber of a map $f\colon X \to Y$ is the homotopy pullback of the diagram $* \to Y \xleftarrow{f} X$.

Proof. $\mathrm{holim}(X \xrightarrow{f} Y \xleftarrow{*} *) = \{(x, *, \gamma) \in X \times * \times \mathrm{Path}(Y) \mid \gamma(0) = f(x), \gamma(1) = *\} = F(f)$. □

3.8.1. **Homotopy limits.** To get homotopy limits, we dualize our work for homotopy colimits. The result is nothing unexpected; see [Str11, §6.6] for a summary.

3.9. (WIP) Rest of homotopy colimits section

3.9.1. **Behavior of colimits under diagram homotopy.** The following shows homotopy equivalent diagrams have homotopy equivalent colimits.

Proposition 3.9.1.1 ([Str11, Proposition 6.2, notation adapted])**.** *Let $\alpha_0, \alpha_1\colon F \to G$ be morphisms of diagrams, and let $H\colon \alpha_0 \simeq \alpha_1$ be a diagram homotopy. Then the maps $f_0, f_1\colon \operatorname{colim} F \to \operatorname{colim} G$ induced respectively by α_0 and α_1 are homotopic in $\mathcal{T}$.*

Proof. By functoriality of the colimit $\operatorname{colim}\colon \mathcal{T}^{\mathcal{J}} \to \mathcal{T}$, the diagram morphisms $\alpha_0, \alpha_1\colon F \to G$ induce maps $f_0 \coloneqq (\alpha_0)_*\colon \operatorname{colim} F \to \operatorname{colim} G$ and $f_1 \coloneqq (\alpha_1)_*\colon \operatorname{colim} F \to \operatorname{colim} G$. Similarly, the diagram morphism $\iota_0, \iota_1\colon F \to \mathrm{Cyl}(F)$ induce maps $\operatorname{colim} F \to \operatorname{colim}(\mathrm{Cyl}(F))$, which by an abuse of notation we will also call ι_0 and ι_1 respectively. Then where $H_*\colon \operatorname{colim}(\mathrm{Cyl}(F)) \to \operatorname{colim} G$ is the map in $\mathcal{T}$ induced by the diagram homotopy H, we have the following commutative diagram in $\mathcal{T}$.

$$\begin{tikzcd}
\operatorname{colim} F \arrow[r, "\iota_0"] \arrow[dr, "f_0"'] & \operatorname{colim}(\mathrm{Cyl}(F)) \arrow[d, "H_*"] & \operatorname{colim} F \arrow[l, "\iota_1"'] \arrow[dl, "f_1"] \\
& \operatorname{colim} G &
\end{tikzcd}$$

Recall that by currying we have an adjunction $\mathrm{Cyl} \dashv \mathrm{Path}$ as functors $\mathcal{T} \to \mathcal{T}$, so in particular Cyl is a left adjoint and hence commutes with colim up to a unique isomorphism. Thus there is a commutative diagram of the above form but with $\operatorname{colim}(\mathrm{Cyl}(F))$ replaced with $\mathrm{Cyl}(\operatorname{colim} F)$. Thus $H_*\colon f_0 \simeq f_1$ in $\mathcal{T}$. ☐

Proposition 3.9.1.2 ([Str11, Proposition 6.5, notation adapted]). *If $\alpha\colon F \to G$ is a diagram homotopy equivalence, then the induced map $f\colon \operatorname{colim} F \to \operatorname{colim} G$ is a homotopy equivalence in $\mathcal{T}$.*

Proof. Since $\alpha\colon F \to G$ is a diagram homotopy equivalence, there is a diagram morphism $\beta\colon G \to F$ and homotopies of diagram morphisms $H\colon \alpha \circ \beta \simeq \mathrm{id}_G$ and $K\colon \beta \circ \alpha \simeq \mathrm{id}_F$. By functoriality of the colimit, $(\alpha \circ \beta)_* = \alpha_*\beta_*$ and $(\beta \circ \alpha)_* = \beta_*\alpha_*$, so by Proposition 3.9.1.1, H and K induces homotopies in $\mathcal{T}$ of the form $\beta_*\alpha_* = (\beta \circ \alpha)_* \simeq (\mathrm{id}_F)_* = \mathrm{id}_{\operatorname{colim} F}$ and $\alpha_*\beta_* = (\alpha \circ \beta)_* \simeq (\mathrm{id}_G)_* = \mathrm{id}_{\operatorname{colim} G}$ respectively, as desired. ☐

3.9.2. **Cofibrant diagrams.** A *pointwise fibration* of diagrams is a diagram morphism $\pi\colon E \to B$ such that $\pi_j\colon E(j) \to B(j)$ is a fibration in $\mathcal{T}$ for all $j \in \mathcal{J}$. A diagram $F \in \mathcal{T}^{\mathcal{J}}$ is *cofibrant* if it is a cofibrant object in the model category $\mathcal{T}^{\mathcal{J}}$, or equivalently, if for any diagram map $\varphi\colon F \to B$ and any pointwise fibration $\pi\colon E \to B$ of diagrams that is also a pointwise homotopy equivalence, there is a lift that makes the following commute in $\mathcal{T}^{\mathcal{J}}$.

$$\begin{tikzcd}
& E \arrow[d, "\pi"] \\
F \arrow[ur, dashed] \arrow[r, "\alpha"] & B
\end{tikzcd} \tag{3.9.2.1}$$

Recall that every map $f \in \mathcal{T}_\circ(X \to Y)$ is pointwise homotopy equivalent to a fibration $p \in \mathcal{T}(E_f \to Y)$ in a functorial way. The proof of that result easily generalizes to the following setting.

Proposition 3.9.2.2 ([Str11, Problem 6.8, notation adapted]). *Any diagram morphism*

$\alpha \in \mathcal{T}^{\mathcal{J}}(F \to G)$ functorially factors as

$$\begin{tikzcd}
F \arrow[rr, "\alpha"] \arrow[dr, "\varepsilon"'] & & G \\
& \widetilde{F} \arrow[ur, "\pi"'] &
\end{tikzcd}$$

where π is a pointwise fibration of diagrams and ε is a diagram homotopy equivalence with diagram homotopy inverse $\overline{\varepsilon}\colon \widetilde{F} \to F$ such that $\overline{\varepsilon} \circ \varepsilon = \mathrm{id}_F$.

Lemma 3.9.2.3. *If $F\colon \mathcal{J} \to \mathcal{T}$ is a cofibrant diagram and $\alpha\colon E \to B$ is a pointwise homotopy equivalence of diagrams, then there is a diagram morphism $\beta\colon F \to E$ making the following commute in $\mathcal{T}^{\mathcal{J}}$ up to diagram homotopy.*

$$\begin{tikzcd}
& E \arrow[d, "\alpha"] \\
F \arrow[ur, dashed, "\exists\,\beta"] \arrow[r] & B
\end{tikzcd} \tag{3.9.2.4}$$

Proof. By Proposition 3.9.2.2, it suffices to show prove the claim for the diagram obtained from (3.9.2.4) by replacing E with $\widetilde{E}$ and α with π where $\widetilde{E}$ is diagram homotopy equivalent to E and π is a pointwise homotopy equivalence of diagrams that is also a pointwise fibration of diagrams. And this is precisely what it means for F to be cofibrant, so we are done. $\square$

Theorem 3.9.2.5 ([Str11, Theorem 6.9, notation adapted])**.** *If F and G are cofibrant diagrams and $\alpha\colon F \to G$ is a pointwise homotopy equivalence of diagrams, then α is a diagram homotopy equivalence.*

Proof. Since G is cofibrant, we can apply Lemma 3.9.2.3 to the following diagram on the left to get a map $\beta\colon G \to F$ and a diagram homotopy $\alpha \circ \beta \simeq \mathrm{id}_G$.

$$\begin{tikzcd}
& F \arrow[d, "\alpha"] \\
G \arrow[ur, dashed, "\exists\,\beta"] \arrow[r, "\mathrm{id}_G"] & G
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& G \arrow[d, "\beta"] \\
F \arrow[ur, dashed, "\exists\,\gamma"] \arrow[r, "\mathrm{id}_F"] & F
\end{tikzcd}$$

Since F is cofibrant, we can apply the same result once more as in the diagram on the right to obtain a map $\gamma\colon G \to F$ and a diagram homotopy $\beta \circ \gamma \simeq \mathrm{id}_F$. But then

$$\beta \circ \alpha = \beta \circ \alpha \circ \underset{F}{\mathrm{id}} \simeq \beta \circ \alpha \circ \beta \circ \gamma \simeq \beta \circ \underset{G}{\mathrm{id}} \circ \gamma = \beta \circ \gamma \simeq \underset{F}{\mathrm{id}}\,. \qquad \square$$

3.9.3. **Cofibrant replacements of diagrams.** If we are given a diagram $F\colon \mathcal{J} \to \mathcal{T}$ that is not cofibrant, then we can hope to replace it with a pointwise homotopy equivalent diagram that is cofibrant. A pointwise homotopy equivalence $\overline{F} \to F$ where $\overline{F}$ is cofibrant is called a *cofibrant replacement* of F. For many index categories cofibrant replacement can be done functorially, but it is often more convenient to use some other ad hoc replacement. For example, if we recognize that a given diagram is already cofibrant, why modify it at all?

Since we will make use of ad hoc cofibrant replacements of diagrams, it will be useful to have answers to the following questions. Does a diagram morphism induce a diagram morphism between their cofibrant replacements? What do we need to know about a functor $\Phi\colon \mathcal{T} \to \mathcal{T}$ to know that the induced functor $\Phi_*\colon \mathcal{T}^{\mathcal{J}} \to \mathcal{T}^{\mathcal{J}}$ carries cofibrant diagrams to cofibrant diagrams?

First we consider lifting a morphism of diagrams to a morphism of cofibrant replacements.

Exercise 3.9.3.1 ([Str11, Problem 6.16])**.** Let $\phi\colon F \to G$ be a diagram morphism, and let $\overline{F} \to F$ and $\overline{G} \to G$ be any two cofibrant replacements.

(a) Show that there is a morphism of diagrams $\overline{\phi}\colon \overline{F} \to \overline{G}$ that is compatible with ϕ in the sense that the following commutes up to diagram homotopy.

$$\begin{tikzcd}
\overline{F} \arrow[r, dashed, "\overline{\phi}"] \arrow[d] & \overline{G} \arrow[d] \\
F \arrow[r, "\phi"] & G
\end{tikzcd} \tag{3.9.3.2}$$

(b) Show that if ϕ is a pointwise homotopy equivalence, then $\overline{\phi}$ is a diagram homotopy equivalence.

(c) Show that if $\overline{F}$ and $\widehat{F}$ are cofibrant replacements for F, then there is a diagram homotopy equivalence $\overline{F} \simeq \widehat{F}$.

Proof. (a): Since $\overline{G} \to G$ is a pointwise homotopy equivalence of diagrams and $\overline{F}$ is cofibrant, the desired solution up to diagram homotopy exists by Lemma 3.9.2.3.

(b): Since ϕ and $\overline{F} \to F$ is a pointwise homotopy equivalence in, so is the composite $\overline{F} \to F \overset{\phi}{\to} G$. By commutativity of (3.9.3.2) up to diagram homotopy, the composite $\overline{F} \overset{\overline{\phi}}{\to} \overline{G} \to G$ is also a pointwise homotopy equivalence. Since homotopy equivalences in $\mathcal{T}$ (and hence also pointwise homotopy equivalences in $\mathcal{T}^{\mathcal{J}}$) satisfy the 2 out of 3 rule, $\overline{\phi}$ is then a pointwise homotopy equivalence. But $\overline{F}$ and $\overline{G}$ are cofibrant, so by Theorem 3.9.2.5 $\overline{\phi}$ is a diagram homotopy equivalence.

(c): This is just (b) when $\phi = \mathrm{id}_F$ and $\overline{G} \to G$ is $\widehat{F} \to F$. □

3.9.4. **Preservation of cofibrant diagrams.** Now we ask which functors preserve cofibrant diagrams.

Theorem 3.9.4.1 ([Str11, Theorem 6.17])**.** *If $L\colon \mathcal{T} \to \mathcal{T}$ is left adjoint to a functor $R\colon \mathcal{T} \to \mathcal{T}$ that preserves fibrations and homotopy equivalences, then $L \circ F$ is a cofibrant diagram whenever F is.*

Proof. Since L is left adjoint to R, the following lifting problems are equivalent in $\mathcal{T}$ in the sense that a solution to one exists if and only if there is a solution to the other.

$$\begin{tikzcd} & A \arrow[d, "f"] \\ L(X) \arrow[ur, dashed] \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} & R(A) \arrow[d, "R(f)"] \\ X \arrow[ur, dashed] \arrow[r] & R(B) \end{tikzcd}$$

Now pass to the diagram category, apply the above equivalence, and use the fact that F is cofibrant to obtain the result. □

Corollary 3.9.4.2 ([Str11, Corollary 6.19])**.**

(a) *If $F\colon \mathcal{J} \to \mathcal{T}_*$ is cofibrant, then so are $F \rtimes A$, $F \wedge A$, and $F \times A$.*

(b) *If $F\colon \mathcal{J} \to \mathcal{T}_\circ$ is cofibrant, then so is $F \times A$.*

3.9.5. **Homotopy colimits of cofibrant diagrams.** We will prove in the next section that for many of the most useful index categories $\mathcal{J}$, essentially every diagram has a functorial cofibrant replacement. We begin by defining the homotopy colimit of a diagram in an ad hoc manner. We will come to the more systematic categorical point of view later.

A space X is a *homotopy colimit* for a diagram $F\colon \mathcal{J} \to \mathcal{T}$, written $X = \operatorname{hocolim} F$, if $X \simeq \operatorname{colim} \overline{F}$ for some cofibrant replacement $\overline{F}$ of F.

Note that the diagram map $\overline{F} \to F$ induces a comparison map $\operatorname{hocolim} F \to \operatorname{colim} F$.

Exercise 3.9.5.1 ([Str11, Problem 6.21])**.**

(a) Show that any two homotopy colimits for a diagram F are homotopy equivalent.

(b) Show that once a cofibrant replacement $\overline{F} \to F$ has been chosen, there are well-defined homotopy classes $F(j) \to \operatorname{hocolim} F$ for each $j \in \mathcal{J}$.

Proof. (a): Suppose we have homotopy equivalences $X \simeq \operatorname{colim} \overline{F}$ and $X' \simeq \operatorname{colim} \widehat{F}$ in $\mathcal{T}$, where $\overline{F} \to F$ and $\widehat{F} \to F$ are cofibrant replacements of F. By Exercise 3.9.3.1, there is a diagram homotopy equivalence $\overline{\phi}\colon \overline{F} \to \widehat{F}$. Thus by [Str11, Proposition 6.5] (Proposition 3.9.1.2), the induced map of colimits $\operatorname{colim} \overline{F} \to \operatorname{colim} \widehat{F}$ is a homotopy equivalence in $\mathcal{T}$. Thus we have homotopy equivalences $X \simeq \operatorname{colim} \overline{F} \simeq \operatorname{colim} \widehat{F} \simeq X'$ in $\mathcal{T}$, as desired. □

Since the homotopy colimit of a diagram is well-defined up to homotopy type, we often speak of *the* homotopy colimit of F.

Exercise 3.9.5.2 ([Str11, Problem 6.22])**.** Compare the homotopy colimit of $* \leftarrow X \to *$ in $\mathcal{T}$ with the colimit of $* \leftarrow X \to *$ in $\mathsf{Ho}\,\mathcal{T}$.

3.9.6. **Induced maps of homotopy colimits.** If $F \to G$ is a diagram morphism, then the formal properties of colimits yield a unique map $\operatorname{colim} F \to \operatorname{colim} G$. By contrast, the homotopy colimit of F is usually not the categorical colimit of F, so a map of diagrams does not in general give a map of homotopy colimits.

Exercise 3.9.3.1 implies a kind of limited naturality to the construction.

Exercise 3.9.6.1 ([Str11, Problem 6.23])**.** Suppose $\phi\colon F \to G$ is a diagram morphism. Show that ϕ induces a map $\Phi\colon \operatorname{hocolim} F \to \operatorname{hocolim} G$ making the following commute up to homotopy in $\mathcal{T}$.

$$\begin{tikzcd}
\operatorname{hocolim} F \arrow[r, dashed, "\Phi"] \arrow[d, "\xi_F"'] & \operatorname{hocolim} G \arrow[d, "\xi_G"] \\
\operatorname{colim} F \arrow[r, "\operatorname{colim} \phi"] & \operatorname{colim} G
\end{tikzcd}$$

Proof. Apply the colimit functor $\operatorname{colim}\colon \mathcal{T}^{\mathcal{J}} \to \mathcal{T}$ to (3.9.3.2) and note that $\operatorname{hocolim} F$ means any space homotopy equivalent to $\operatorname{colim} \overline{F}$, so the given diagram is only guaranteed to commute up to homotopy. ☐

The above map Φ is not uniquely determined by ϕ, even up to homotopy. We should think of a map of diagrams ϕ as inducing a set of maps between homotopy colimits rather than a single map. However, these maps are closely related to one another.

Exercise 3.9.6.2 ([Str11, Problem 6.24])**.** Let $\phi\colon F \to G$ be a diagram map and choose cofibrant replacements $\overline{F} \to F$, $\widehat{F} \to F$, $\overline{G} \to G$, and $\widehat{G} \to G$.

(a) Show the following commutes up to diagram homotopy in $\mathcal{T}^{\mathcal{J}}$.

$$\begin{tikzcd}
\widehat{F} \arrow[rrr, "\widehat{\phi}"] \arrow[dr] \arrow[dd, "\simeq"'] & & & \widehat{G} \arrow[dl] \arrow[dd, "\simeq"] \\
& F \arrow[r, "\phi"] & G & \\
\overline{F} \arrow[rrr, "\overline{\phi}"] \arrow[ur] & & & \overline{G} \arrow[ul]
\end{tikzcd}$$

(b) Conclude that any two induced maps of homotopy colimits are pointwise equivalent in $\mathsf{Ho}\,\mathcal{T}$, i.e., ϕ_j is a homotopy equivalence for each $j \in \mathcal{J}$ and for all $k \in \mathcal{J}(j \to j')$ and the following commutes up to homotopy in $\mathcal{T}$.

$$\begin{tikzcd}
F(j) \arrow[r, "\phi_j"] \arrow[d, "F(k)"'] & G(j) \arrow[d, "G(k)"] \\
F(j') \arrow[r, "\phi_{j'}"] & G(j')
\end{tikzcd}$$

3.9.7. **Preservation of homotopy colimits by functors.** See [Str11, Theorem 6.94] or more generally [Str11, §6.8].

3.9.8. **Functorial approach to homotopy colimits.** Here we construct a homotopy colimit *functor* $\operatorname{hocolim}\colon \mathcal{T}^{\mathcal{J}} \to \mathcal{T}$. Suppose we have constructed a functor $\operatorname{Cof}\colon \mathcal{T}^{\mathcal{J}} \to \mathcal{T}^{\mathcal{J}}$ and a natural transformation $\xi\colon \operatorname{Cof} \Rightarrow \mathrm{id}_{\mathcal{T}^{\mathcal{J}}}$ such that for all diagrams $F \in \mathcal{T}^{\mathcal{J}}$ the map $\xi_F\colon \operatorname{Cof}(F) \to F$ in $\mathcal{T}$ is a cofibrant replacement. Then define the *homotopy colimit functor* $\operatorname{hocolim}\colon \mathcal{T}^{\mathcal{J}} \to \mathcal{T}$ by $\operatorname{hocolim} := \operatorname{colim} \circ \operatorname{Cof}$.

Theorem 3.9.8.1 ([Str11, Theorem 6.27])**.**

(a) *If $F \to G$ is a pointwise homotopy equivalence of diagrams, then the induced map* $\operatorname{hocolim} F \to \operatorname{hocolim} G$ *is a homotopy equivalence in $\mathcal{T}$.*

(b) *If $\mathcal{L}\colon \mathcal{T}^{\mathcal{J}} \to \mathcal{T}$ satisfies the property of* hocolim *in (a) and $\zeta\colon \mathcal{L} \Rightarrow \operatorname{colim}$ is any natural transformation, then there is a unique natural transformation* $\mathsf{Ho} \circ \mathcal{L} \Rightarrow \mathsf{Ho} \circ \operatorname{hocolim}$ *such that the following diagram commutes in* $(\mathsf{Ho}\,\mathcal{T})^{\mathcal{J}}$.

$$\begin{tikzcd}
 & \mathsf{Ho} \circ \operatorname{hocolim} \arrow[d, "\xi"] \\
\mathsf{Ho} \circ \mathcal{L} \arrow[ur, dashed, "\exists!"] \arrow[r, "\mathsf{Ho}(\zeta)"] & \mathsf{Ho} \circ \operatorname{colim}
\end{tikzcd}$$

Proof. (a): This is exactly the proof of Exercise 3.9.3.1(b).

(b): First fix a diagram $F\colon \mathcal{J} \to \mathcal{T}$. Since ζ is a natural transformation, the the following diagram commutes in $\mathcal{T}$.

$$\begin{tikzcd}
\mathcal{L}(\operatorname{Cof}(F)) \arrow[r, "\zeta_{\operatorname{Cof}(F)}"] \arrow[d, "\mathcal{L}(\xi_F)"'] & \operatorname{colim}(\operatorname{Cof}(F)) \arrow[d, "\operatorname{colim}(\xi_F)"] \\
\mathcal{L}(F) \arrow[r, "\zeta_F"] & \operatorname{colim}(F)
\end{tikzcd}$$

Since $\xi_F\colon \operatorname{Cof}(F) \to F$ is a cofibrant replacement, it is a pointwise homotopy equivalence. Since $\mathcal{L}$ satisfies the condition in (a) for hocolim, $\mathcal{L}(\xi_F)$ is a homotopy equivalence in $\mathcal{T}$. Where $\mathcal{L}(F)^{-1}$ is the diagram homotopy inverse to $\mathcal{L}(F)$, applying Ho to the above diagram, the desired lift is $\mathsf{Ho}(\mathcal{L}(F)) \xrightarrow{\mathsf{Ho}(\mathcal{L}(\xi_F)^{-1})} \mathcal{L}(\operatorname{Cof}(F)) \xrightarrow{\zeta_{\operatorname{Cof}(F)}} \operatorname{colim}(F)$. And this lift is unique, since any other lift $\eta\colon \mathsf{Ho} \circ \mathcal{L} \Rightarrow \mathsf{Ho} \circ \operatorname{hocolim}$ must satisfy $\eta_F \circ \mathcal{L}(\xi_F) = \zeta_{\operatorname{Cof}(F)}$; by applying the unique inverse to $\mathcal{L}(\xi_F)$ in $\mathsf{Ho}\,\mathcal{T}$ to this equation, η is given by the same formula. ☐

We call $\operatorname{hocolim} F := \operatorname{colim}(\operatorname{Cof}(F))$ the *standard homotopy colimit* to distinguish it from the “ad hoc” homotopy colimit defined before.

3.9.9. **Construction of the cofibrant replacement functor for simple diagrams.** Now that we know how to recognize a cofibrant diagram, we are equipped to build functorial cofibrant replacements.

Theorem 3.9.9.1 ([Str11, Theorem 6.45])**.** *If $\mathcal{J}$ is a simple category, then there is a functor* $\mathrm{Cof}\colon \mathcal{T}^{\mathcal{J}} \to \mathcal{T}^{\mathcal{J}}$ *and a natural transformation* $\xi\colon \mathrm{Cof} \Rightarrow \mathrm{id}_{\mathcal{T}^{\mathcal{J}}}$ *that assigns to every diagram* $F\colon \mathcal{J} \to \mathcal{T}$ *(of well-pointed spaces when* $\mathcal{T} = \mathcal{T}_*$*) a cofibrant replacement* $\xi_F\colon \overline{F} \to F$ *in* $\mathcal{T}$*.*

The idea of the proof is to define cofibrant replacements $\overline{F}_n\colon \mathcal{J}_n \to \mathcal{T}$ for F_n inductively, starting with the identity transformation $\mathrm{id}_{F_0}\colon F_0 = \overline{F}_0 \to F_0$. Suppose now that a pointwise homotopy equivalence $\overline{F}_n \to F_n$ has been defined so that $\overline{F}_n$ satisfies the conditions of Theorem 3.7.5.1. For $j \in \mathcal{J}_{n+1} \smallsetminus \mathcal{J}_n$, define $\overline{F}_{n+1}(j)$ to be the mapping cylinder M_{ξ_j} in $\mathcal{T}$ of the comparison map $\xi_j\colon \operatorname{colim} \overline{F}_{<j} \to F(j)$.

Exercise 3.9.9.2 ([Str11, Problem 6.46])**.** Finish the construction of the functor $\overline{F}$ and the transformation $\overline{F} \to F$, and use it to prove Theorem 3.9.9.1.

3.10. More homotopy groups

3.10.1. **LES of relative homotopy groups induced by a pair.** The following is an excerpt that makes up [May11, §§9.1–2, heavily adapted], with the notation adapted for our purposes. Consider a pair (A, X) with inclusion $i\colon A \to X$. For $F(i) \coloneqq P(X; *, A)$, we have the long fiber sequence

$$\cdots \longrightarrow \Omega^2 A \longrightarrow \Omega^2 X \longrightarrow \Omega(F(i)) \longrightarrow \Omega A \longrightarrow \Omega X \longrightarrow F(i) \xrightarrow{p_1} A \xrightarrow{i} X$$

associated to the inclusion $i : A \to X$, where p_1 is the endpoint projection and ι is the inclusion. Applying the functor $\pi_0 = [S^0, -]$ to this sequence and judiciously applying the suspension-loop space adjunction, we obtain the long exact sequence

$$\cdots \longrightarrow \pi_n(A) \longrightarrow \pi_n(X) \longrightarrow \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \longrightarrow \cdots \longrightarrow \pi_0(A) \longrightarrow \pi_0(X).$$

Using $\pi_n(X, A, *) = [(I^n, \partial I^n, J^n), (X, A, *)]$, the map $\partial : \pi_n(X, A) \to \pi_{n-1}(A)$ is obtained by restricting maps $(I^n, \partial I^n, J^n) \to (X, A, *)$ to maps $(I^{n-1} \times \{1\}, \partial I^{n-1} \times \{1\}) \to (A, *)$ while $\pi_n(A) \to \pi_n(X)$ and $\pi_n(X) \to \pi_n(X, A)$ are induced by the inclusions $(A, *) \hookrightarrow (X, *)$ and $(X, *, *) \hookrightarrow (X, A, *)$ respectively.

3.10.2. **LES of homotopy groups induced by a fibration.** The following is an excerpt that makes up [May11, §9.3], with the notation adapted for our purposes.

Let $p\colon E \to B$ be a fibration over a path-connected space B. Fix a basepoint $* \in B$, let $F = p^{-1}(*)$, and fix a basepoint $* \in F \subseteq E$. The inclusion $\phi : F \to F(p)$ is a homotopy equivalence, and, being pedantically careful to choose signs appropriately, we obtain the following diagram, in which two out of each three consecutive squares commute and the third commutes up to homotopy.

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega^2 E \arrow[r, "-\Omega\iota"] \arrow[d, "\mathrm{id}"] & \Omega F(i) \arrow[r, "-\Omega p_1"] \arrow[d, "\Omega\phi"] & \Omega F \arrow[r, "-\Omega i"] \arrow[d, "\Omega\phi"] & \Omega E \arrow[r, "\iota"] \arrow[d, "\mathrm{id}"] & F(i) \arrow[r, "p_1"] \arrow[d, "-p"] & F \arrow[r, "i"] \arrow[d, "\phi"] & E \arrow[d, "\mathrm{id}"] \\
\cdots \arrow[r] & \Omega^2 E \arrow[r, "\Omega^2 p"'] & \Omega^2 B \arrow[r, "-\Omega\iota"'] & \Omega F(p) \arrow[r, "-\Omega\pi"'] & \Omega E \arrow[r, "-\Omega p"'] & \Omega B \arrow[r, "\iota"'] & F(p) \arrow[r, "\pi"'] & E
\end{tikzcd}$$

Here $F(i) = P(E; *, F)$, $p(\xi) = p \circ \xi \in \Omega B$ for $\xi \in F(i)$, and the next to last square commutes up to the homotopy $h : \iota \circ (-p) \simeq \phi \circ p_1$ specified by $h(\xi, t) = (\xi(t), p(\xi[1, t]))$, where $\xi[1, t](s) = \xi(1 - s + st)$. Passing to long exact sequences of homotopy groups and using the five lemma, together with a little extra argument in the case $n = 1$, we conclude that $p_* : \pi_n(E, F) \longrightarrow \pi_n(B)$ is an isomorphism for $n \geq 1$.

Using ϕ_* to identify $\pi_* F$ with $\pi_*(F(p))$, we may rewrite the long exact sequence of the bottom row of the diagram as

$$\cdots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \longrightarrow \cdots \longrightarrow \pi_0(E) \longrightarrow \{*\}.$$

(At the end, a little path lifting argument shows that $\pi_0(F) \longrightarrow \pi_0(E)$ is a surjection.) This is one of the main tools for the computation of homotopy groups.

3.10.3. ***n*-equivalences and weak equivalences.** A map $e\colon Y \to Z$ is an *n-equivalence* if for all $y \in Y$ the map $e_*\colon \pi_q(Y, y) \to \pi_q(Z, e(y))$ is injective for $q \leq n - 1$ and surjective for $q \leq n$. We call e a *weak equivalence* if it is an n-equivalence for all n, or equivalently if $e\colon X \to Y$ induces isomorphisms $e_*\colon \pi_n(X) \xrightarrow{\cong} \pi_n(Y)$ on all homotopy groups. We call spaces $X, Y \in \mathcal{T}$ *weakly equivalent* if there is a weak equivalence $X \to Y$. The *homotopy type* of X is the data $\{\pi_n(X)\}_{n \geq 0}$ of the homotopy groups of X, and thus weak homotopy equivalences are precisely the maps that preserve homotopy type. Thus any homotopy equivalence is a weak equivalence, but note that the converse is false.

3.10.4. **Actions of the fundamental groupoid on fibers.** In the theory of covering spaces, the fundamental group $\pi_1(B, b_0)$ of the base space of a covering $p\colon E \to B$ induces homeomorphisms (i.e., deck transformations, i.e., covering automorphisms) of the fibers $F_b \coloneqq p^{-1}(b_0)$ over B. If instead of a covering map we have a fibration $p\colon E \to B$, then it turns out a similar statement is true: the fundamental group $\pi_1(B, b_0)$ of the base space of a fibration acts on the fiber, though only up to homotopy equivalence. We prove this here.

3.10.4.1. *The fundamental groupoid of a space.* Define the *fundamental groupoid* $\Pi_1(X)$ to be the category (in fact, groupoid) whose objects are the points $x \in X$ and whose morphisms in $\Pi_1(X)(x \to x')$ are homotopy classes of continuous paths in X from x to x'; thus $\pi_1(X, x_0) = \Pi_1(X)(x_0 \to x_0)$. As is usual, we tend identify groupoids with their morphisms rather than their objects. **TODO: Mention how the nth fundamental groupoid is an n-category**

3.10.4.2. *Homotopy equivalence of fibers induced by paths in the base space of fibrations.* Here we follow the beginning of [May11, Ch. 7, §4]. Let $p\colon E \to B$ be a fibration, and for each $b \in B$, let $F_b \overset{i_b}{\hookrightarrow} E \overset{p}{\to} B$ be the corresponding strict fiber sequence, so that $F_b = p^{-1}(b)$. A path $\gamma\colon I \to B$ induces a homotopy equivalence on the fibers $F_{\gamma(0)}$ and $F_{\gamma(1)}$. Indeed, since p is a fibration, there is a lift $h_\bullet\colon F_b \times I \to E$ as in the following diagram on the left.

$$\begin{tikzcd}
F_{\gamma(0)} \arrow[r, hook] \arrow[d, hook] & E \arrow[d, "p"] \\
F_{\gamma(0)} \times I \arrow[r, "\gamma\circ\mathrm{pr}_2"] \arrow[ur, dashed, "h_\bullet"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
F_{\gamma(1)} \arrow[r, hook] \arrow[d, hook] & E \arrow[d, "p"] \\
F_{\gamma(1)} \times I \arrow[r, "\overline{\gamma}\circ\mathrm{pr}_2"] \arrow[ur, dashed, "h'_\bullet"] & B
\end{tikzcd}$$

Thus, by commutativity, $h_1\colon F_{\gamma(0)} \times \{1\} \to E$ satisfies $\{\gamma(1)\} = \mathrm{im}(p \circ h_1)$, so $F_{\gamma(1)} = p^{-1}(\gamma(1)) = p^{-1}(\mathrm{im}(p \circ h_1)) \supseteq \mathrm{im}\, h_1$. Thus we can view h_1 as a map $F_{\gamma(0)} \to F_{\gamma(1)}$. We could have instead started with the above square but with γ replaced with its reversal $\overline{\gamma}$, which would result in a map $h'_1\colon F_{\gamma(1)} \to F_{\gamma(0)}$, and one can check that $h'_1 \circ h_1 \simeq \mathrm{id}_{F_{\gamma(0)}}$ and $h_1 \circ h'_1 \simeq \mathrm{id}_{F_{\gamma(1)}}$, which shows $F_{\gamma(0)} \simeq F_{\gamma(1)}$.

3.10.4.3. *Action of the fundamental groupoid on fibers of a fibration.* Moreover, one can show the above homotopy equivalence is the same for any choice of representative in the homotopy class of γ; see [May11, §7.4, pp. 53–4]. This means each homotopy class $[\gamma] \in \Pi_1(B)(b_0 \to b_1)$ induces a homotopy equivalence $F_{b_0} \overset{\simeq}{\to} F_{b_1}$, which gives a homotopy action of $\Pi_1(B)$ on the fibers of p. This is often referred to as the *action of the fundamental groupoid on the fibers* or as *translation of fibers along a path.* If $b_1 = b_0$, this means we get an action by the fundamental group $\pi_1(B, b_0)$ on the homotopy classes of homotopy equivalences of F_{b_0} with itself, which we denote by $\mathrm{Aut}_{\mathsf{Ho}\mathcal{T}}(F_{b_0})$. One can check this gives a group homomorphism $\pi_1(B, b_0) \to \mathrm{Aut}_{\mathsf{Ho}\mathcal{T}}(F_{b_0})$.

3.10.5. **Action of the fundamental group on higher homotopy groups.** Here we follow §4 of these notes. For $X \in \mathcal{T}_\circ$, $x_0 \in X$, and a homotopy $H\colon S^n \times I \to X$ between two representatives $f, g\colon S^n \to X$ of two homotopy classes $[f], [g] \in \pi_n(X, x_0)$, we get a *path* $\gamma\colon I \to X$ given by $\gamma(t) \coloneqq H_t(\mathbf{1})$, where we recall $\mathbf{1}$ is the basepoint of S^n. Since H is not a pointed map, γ is possibly nonconstant. We write $f \simeq_\gamma g$ to indicate $\gamma \subseteq X$ is a path obtained in this way.

Let $\overline{\gamma}$ and $\gamma * \xi$ denote the reversal of the path γ and the concatenation of γ followed by ξ, respectively. One uses that the homotopy relation is an equivalence relation to show (i) $f \simeq_\gamma g \iff g \simeq_{\overline{\gamma}} f$ and (ii) $f \simeq_\gamma g \simeq_\xi h \iff f \simeq_{\xi * \gamma} gh$. In addition, one can show (iii) if $\gamma \simeq \xi \operatorname{rel} \partial I$ and $f \simeq_\gamma g$, then $f \simeq_\xi g$, and (iv) if for $x_1 \in X$ we have $f_0(\mathbf{1}) = f_1(\mathbf{1}) = x_0$, $g_0(\mathbf{1}) = g_1(\mathbf{1}) = x_1$, $f_0 \simeq f_1 \operatorname{rel} x_0$, $g_0 \simeq g_1 \operatorname{rel} x_1$, and $f_0 \simeq_\gamma g_0$, then $f_1 \simeq_\xi g_1$. Thus there is a unique homotopy class, denoted ${}^{[\gamma]}[f]$, of maps $g\colon (S^n, \mathbf{1}) \to (X, x_1)$ such that $f \simeq_\gamma g$.

Given pointed maps $f\colon (S^n, \mathbf{1}) \to (X, x_0)$ and $g\colon (S^n, \mathbf{1}) \to (X, x_1)$ and a homotopy $H\colon f \simeq_\gamma g$ (so $\gamma\colon I \to X$ has $\gamma(0) = x_0$ and $\gamma(1) = x_1$, $[g] \in \pi_n(X, x_1)$ is completely determined by the pair $([\gamma], [f])$. Indeed, if $f \simeq_{c_{x_0}} f'$, $\gamma \simeq \xi \operatorname{rel} \partial I$, and $f' \simeq_\xi g'$, then $g \simeq_{c_{x_1}} g'$ since $g \simeq_{\overline{\gamma}} f \simeq_{c_{x_0}} f' \simeq_\xi g'$ and $\xi * c_{x_0} * \overline{\gamma} \simeq c_{x_1} \operatorname{rel} \partial I$, so our previous observations imply $g \simeq_\gamma g'$.

The previous paragraph describes an action of $\Pi_1(X)$ on higher homotopy groups in the sense that we have a pairing

$$\begin{aligned} \Pi_1(X)(x_0 \to x_1) \times \pi_n(X, x_0) &\longrightarrow \pi_n(X, x_1), \\ ([\gamma], [f]) &\longmapsto {}^{[\gamma]}[f] \end{aligned} \tag{3.10.5.1}$$

inducing functors $\pi_n(X, -)\colon \Pi_1(X) \to \mathsf{Grp}$ given by $x_0 \mapsto \pi_n(X, x_0)$ and $[\gamma] \mapsto {}^{[\gamma]}(-)$. By setting $x_1 = x_0$, we obtain the following.

Corollary 3.10.5.2. *For a path-connected $X \in \mathcal{T}_\circ$ and $x_0 \in X$, the forgetful map $\pi_n(X, x_0) = [(S^n, \mathbf{1}), (X, x_0)] \to [S^n, X]$ gives a bijection of $[S^n, X]$ with the space $\pi_n(X, x_0)/\pi_1(X, x_0)$ of orbits of the action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ given above.*

Observe that whenever x_0 and x_1 lie in the same path component of X, we have an isomorphism ${}^{[\gamma]}(-)\colon \pi_n(X, x_0) \xrightarrow{\cong} \pi_n(X, x_1)$, where γ is any path from x_0 to x_1. This isomorphism is canonical precisely when $\pi_1(X, x_0) = \{1\}$. When $n = 1$, this action is just conjugation of the path $[f]$ by $[\gamma]$.

3.10.5.1. *Another exposition of the action of π_1 on π_n.* Here we use §1 of here. See also these notes. We write $\pi_n(X)$ for $\pi_n(X, *)$ in (1.1.10.1) because $\pi_n(X)$ is, up to isomorphism, independent of the basepoint $* \in X$, as we now show.

Proposition 3.10.5.3. *For $n \geq 1$ and path-connected $X \in \mathcal{T}_*$, there is an isomorphism $\beta_\gamma\colon \pi_n(X, x_1) \to \pi_n(X, x_0)$ given by $\beta_\gamma([f]) := [\gamma \cdot f]$, where γ is any path from x_1 to x_0 and $\gamma \cdot f$ is obtained by restricting a representative $f\colon I^n \to X$ of a homotopy class in $\pi_n(X, x_1)$ to a smaller concentric cube inside I^n, and then inserting the path γ on each radial segment of the shell between this smaller cube and ∂I^n.*

Proof. See here, Proposition 1.13. ☐

When we take $x_1 := x_0$, this assignment gives an action of $\pi_1(X)$ on $\pi_n(X)$ by $\pi_1 \times \pi_n \to \pi_n$, $([\gamma], [f]) \mapsto [\gamma \cdot f]$. For example, when $n = 1$, this is just conjugation, i.e., $[\gamma] \cdot [\xi] = [\overline{\gamma} * \xi * \gamma]$.

3.10.5.2. *Abelian spaces and simple spaces.* We call $X \in \mathcal{T}$ an *abelian space* if this action by $\pi_1(X)$ on $\pi_n(X)$ is trivial for all $n \geq 1$; for abelian spaces X, $\pi_1(X)$ is abelian.

For maps $\phi\colon X \to Y$ in $\mathcal{T}$, define $\pi_n(\phi) \coloneqq \phi_*[f]$, so that $\phi_*[f] = [\phi \circ f]$. Since $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_n(X)}$, this makes each π_n a functor $\mathcal{T} \to \mathsf{Grp}$. In fact, when $\phi\colon X \to Y$ is a homotopy equivalence, the induced map is an isomorphism of groups; thus π_n maps $\mathsf{Ho}\,\mathcal{T}$ into Grp. For example, since $\mathbb{R}^n$ is homotopy equivalent to a point, $\pi_n(\mathbb{R}^n) \cong 0$ for all $n \geq 1$.

We call $X \in \mathcal{T}$ *n-simple* if its fundamental group $\pi_1(X)$ is abelian and acts trivially on $\pi_q(X)$ for all $q \in \{0, 1, \dots, n\}$; we call X *simple* if X is n-simple for all $n \geq 0$. For example, simply connected spaces are simple since their fundamental groups are trivial.

The condition of being n-simple is not too restrictive. Indeed, all topological groups, Eilenberg–MacLane spaces, and universal covers—if they have abelian fundamental group—are simple.

Note that the quotient $\pi_n(X)/\pi_1(X)$ of $\pi_n(X)$ by the group action of π_1 is precisely $[S^n, X]$ (which may sometimes be confused with the pointed homotopy classes $[(S^n, \mathbf{1}), (X, *)]$). Thus, if X is n-simple, then $\pi_n(X) \cong [S^n, X]$.

3.10.6. **Obstruction theory.** The first part follows the first few pages of these notes, while the last part follows [May11, §18.1]. Suppose that X and Y are simply connected spaces and $f\colon X \to Y$ is a map that induces isomorphisms for all $\mathbb{Z}$-homology. Prove that f is a weak homotopy equivalence. One reason one is interested in (co)homology with coefficients is that the cohomology groups of a space X with coefficients in the homotopy groups of a space Y control the construction of homotopy classes of maps $X \to Y$. Here we will explain why.

Let (X, A) be a relative CW complex with relative skeleta X^n and let Y be an n-simple space, i.e., $\pi_1(X) \cong [S^n, X]$. Fix a map $f\colon X^n \to Y$. When can f be extended to a map $f\colon X^{n+1} \to Y$ restricting to the given map on A?

We first discuss when f can be extended to a single $(n+1)$-cell in X^n, say indexed by $\alpha \in \Sigma_{n+1}(X)$. To say f extends from the boundary of an $(n+1)$-cell of X to a map on the whole $(n+1)$-cell means that the map $f \circ \varphi_\alpha^{n+1}\colon S_\alpha^n \to Y$ extends to a map $D_\alpha^{n+1} \cong CS_\alpha^n \to Y$. By the characterization of nullhomotopy as factoring through the cone over the domain, this means $f\colon X^n \to Y$ can be extended to an $(n+1)$-cell e_α^{n+1} if and only if $f \circ \varphi_\alpha^n$ is nullhomotopic, i.e., if and only if $[f \circ \varphi_a^{n+1}] = 0$ in $\pi_n(Y)$.

The previous paragraph describes the assignment $\Sigma_{n+1}(X) \to \pi_n(Y)$ given by $\alpha \mapsto [f \circ \varphi_\alpha^n]$. When $n \geq 2$ the target $\pi_n(Y)$ is abelian (i.e., a $\mathbb{Z}$-module), so this set map linearly extends to a $\mathbb{Z}$-module map $C_{n+1}^{\mathrm{CW}}(X, A; \pi_n(Y)) \to \pi_n(Y)$. As the collection of such homomorphisms is $C_{\mathrm{CW}}^{n+1}(X, A; \pi_n(Y))$, the upshot is that there is an $(n+1)$-cochain $\theta_f^{n+1} \in C_{\mathrm{CW}}^n(X, A; \pi_n(Y))$ such that $\theta_f = 0$ if and only if f extends from a map on X^n to a map on X^{n+1}. For this reason we call $\theta^{n+1}(f)$ the *obstruction cochain.*

Remarkably, $\theta^{n+1}(f)$ is a *cocycle*, and thus defines an element $[\theta^{n+1}_f] \in H^{n+1}_{\mathrm{CW}}(X;\pi_n(Y)) \cong H^{n+1}(X;\pi_n(Y))$, called the *obstruction cocycle*. We prove this as follows. Suppose $f, f'\colon X^n \to Y$ and $H\colon f|_{X^{n-1}} \simeq f'|_{X^{n-1}} \operatorname{rel} A$. These define a map $H(f,f')\colon (X\times I)^n \to Y$. Applying $\theta^{n+1}(H(f,f'))$ to cells $e^n_\alpha \times I$, we obtain a cochain $d_{f,f',H}\colon C^n_{\mathrm{CW}}(X,A;\pi_n(Y))$ such that $\delta d_{f,f',H} = \theta^{n+1}(f) - \theta^{n+1}(f')$, called a *deformation cochain*. Given f and d, there exists f' that coincides with f on X^{n-1} and satisfies $d_{f,f'} = d$, where the constant homotopy H is understood. This gives the following result.

Theorem 3.10.6.1. *For a CW complex X and an n-simple space Y, a map $f\colon X^n \to Y$, $f|_{X^{n-1}}$ extends to a map $X^{n+1} \to Y$ if and only if $[\theta^{n+1}(f)] = 0$ in $H^{n+1}(X,A;\pi_n(Y))$.*

It is natural to ask further when such extensions are unique up to homotopy, and a similar argument gives the answer in the form of the following result.

Theorem 3.10.6.2. *Given maps $f,f'\colon X^n \to Y$ and a homotopy $H\colon f|_{X^{n-1}} \simeq f'|_{X^{n-1}} \operatorname{rel} A$, there is an obstruction class $[\theta] \in H^n(X,A;\pi_n(Y))$ that vanishes if and only if $H|_{X^{n-2}\times I}$ extends to a homotopy $f \simeq f' \operatorname{rel} A$.*

3.10.7. **Hurewicz's theorem.** For each $n \geq 0$, choose generators $i_n\colon S^n \to X$ of $\pi_n(S^n) \cong \mathbb{Z}$ such that $\Sigma i_n = i_{n+1}$. For a path-connected space $X \in \mathcal{T}_*$ and an integer $n \geq 0$, the *Hurewicz map* $h\colon \pi_n(X) \to \widetilde{H}_n(X)$ is defined by $h([f]) \coloneqq f_*[[i_n]]$ where $[-]$ and $[[-]]$ denote the homotopy and homology class respectively.

Theorem 3.10.7.1 (Hurewicz's theorem)**.** *Fix $n \in \mathbb{Z}_{\geq 1}$ and a path-connected space $X \in \mathcal{T}_*$.*

(i) *The Hurewicz map is a homomorphism that is natural in the sense that when $\Sigma i_n = i_{n+1}$, the following diagram commutes for all $n \geq 0$.*

$$\begin{tikzcd}
\pi_n(X) \arrow[r, "h"] \arrow[d, "\Sigma"] & \widetilde{H}_n(X) \arrow[d, "\Sigma"] \\
\pi_{n+1}(\Sigma X) \arrow[r, "h"] & \widetilde{H}_{n+1}(\Sigma X)
\end{tikzcd}$$

(ii) *The Hurewicz map $h_1\colon \pi_1 X \to H_1 X$ descends to an isomorphism $\overline{h}_1\colon (\pi_1 X)^{\mathrm{ab}} \xrightarrow{\cong} H_1 X$, where $(\pi_1 X)^{\mathrm{ab}}$ denotes the* abelianization *of $\pi_1 X$, i.e., $\pi_1 X/[\pi_1 X, \pi_1 X]$.*

(iii) *If $n \geq 2$ and $\pi_q X = 0$ for all $q \leq n-1$, then $\widetilde{H}_q X = 0$ for all $1 \leq q \leq n-1$ and the Hurewicz map is an isomorphism $h_n\colon \pi_n X \xrightarrow{\cong} H_n X$.*

For a proof, see [May11, Theorem 15.1]

3.10.8. **Killing prescribed homotopy groups.**

Lemma 3.10.8.1. *For $X \in \mathsf{Top}_*$, attaching an $(n+1)$-cell to get $X' \in \mathsf{Top}_*$ in the sense that we have a pushout*

$$\begin{tikzcd} S^n \arrow[r, hook, "j"] \arrow[d, "f"'] & D^{n+1} \arrow[d, "\Phi"] \\ X \arrow[r, hook, "i"] \arrow[ur, phantom, "\ulcorner", very near end] & X' \end{tikzcd}$$

makes $i_*[f] = 0$. Thus, if we consider the space $\widetilde{X}$ obtained from X by attaching $(n+1)$-cells as above for each nontrivial homotopy class $[f] \in \pi_n X$, then $\pi_q \widetilde{X} = \delta_{q \neq n} \pi_q X$, i.e., $\pi_q \widetilde{X} = \pi_q X$ if $q \neq n$ and $\pi_q \widetilde{X} = 0$ if $q = n$.

Proof. Since D^n is contractible, $j \simeq *$. Thus $\Phi \circ j \simeq \Phi \circ * \simeq *$, so $i_*[f] = i \circ f = \Phi \circ j \simeq *$, i.e., $i_*[f] \simeq *$ as a map on X', as claimed. ☐

3.10.9. **Eilenberg–MacLane spaces and Postnikov towers.** Fix an abelian group π and an integer $n \geq 1$. The following lemma says there exists a unique $X \in \mathsf{Ho}\,\mathcal{T}$ up to canonical isomorphism with the property that $\pi_i X = \delta_{i=n}\pi$, i.e., that $\pi_i X = \pi$ if $i = n$ and $\pi_i X = 0$ otherwise. Such a space is called an *Eilenberg–MacLane space*, or a *$K(\pi, n)$-space*. By its uniqueness up to homotopy equivalence, we often write $K(\pi, n)$ to mean any such space.

Lemma 3.10.9.1. *There is a pointed CW complex X such that $\pi_i(X) \cong \pi$ for $i = n$ and $\pi_i(X) = 0$ otherwise. And for any other such space X', there is a homotopy equivalence $f\colon X \simeq X'$ inducing isomorphisms $\pi_i(X) \to \pi_i(X')$, and any two such equivalences f are homotopic.*

Proof. First consider the Moore space $M(\pi, n)$. Since $M(\pi, n)$ is a wedge sum of S^n with cells of dimension $\geq n+1$ added, $\pi_q(M(\pi, n)) = 0$ for all $q \leq n-1$, so an application of Hurewicz's theorem followed by the universal coefficient theorem implies that $\pi_q(M(\pi, n)) \cong H_q(M(\pi, n); \pi) = 0$ for all $q \leq n-1$.

Now set $X_n := M(\pi, n)$ and for $i \geq 1$ let X_{n+i} be the space obtained from X_{n+i-1} by attaching an $(n+i+1)$-cell for every nontrivial homotopy class $[f] \in \pi_{n+i+1}(X)$, so that by Lemma 3.10.8.1 we have $\pi_{n+i} X_{n+i} = 0$. Then define

$$K(\pi, n) := \operatorname{colim}\left(X_n \hookrightarrow X_{n+1} \hookrightarrow X_{n+2} \hookrightarrow \cdots\right).$$

Then one can show that $\pi_q(K(\pi, n)) = \delta_{q=n}\pi$.

Now suppose W is another connected CW complex such that $\pi_q W = \delta_{q=n}\pi$. Then we have a cone under $X_1 \hookrightarrow X_2 \hookrightarrow \cdots$, say with component morphisms $\lambda_j\colon X_j \to W$, where λ_{n+1} sends an $(n+i)$-cell to the corresponding cell in W. The maps λ_{n+1} are each well-defined by the pasting lemma, and of course they make the obvious diagram commute, so by the universal property of the colimit there is a unique map $K(\pi, n) \to W$ compatible with the cone and the aforementioned diagram. To see that map is a homotopy equivalence, note that W is a CW complex, and hence a colimit of the sequence $W^0 \hookrightarrow W^1 \hookrightarrow \cdots$, and again define the component morphisms of the cone in the obvious way, so similarly we get a unique map $W \to K(n, \pi)$. By composing and using uniqueness, we are done. □

3.10.9.1. *Postnikov towers.* Here we follow the beginning of [Str11, §16.4]. Recall that to prove the CW approximation theorem, we repeatedly attached new cells to construct a CW complex weakly equivalent to a given space. Here we will use the same fundamental construction, but this time, we will take the resulting space and kill off its higher homotopy groups above a prescribed dimension, leaving us with trivial higher homotopy groups thereafter. This construction produces spaces to which we can apply our obstruction theory.

Let X be any space, and let $n \geq 1$. A map $\alpha_n \colon X \to P_n(X)$ is called an *nth Postnikov approximation* if

- $(\alpha_n)_* \colon \pi_k X \to \pi_k(P_n(X))$ is an isomorphism for $k \leq n$, and
- $\pi_k(P_n(X)) = 0$ for $k \geq n + 1$.

Theorem 3.10.9.2 ([Str11, Theorem 16.33])**.** *For any space X and any $n \geq 1$, there is an nth Postnikov approximation $\alpha_n \colon X \to P_n(X)$, and this is unique up to pointwise weak homotopy equivalence of maps.*

Proof. We follow the guide in [Str11, Problem 16.34]. We first prove the existence by inductively constructing a sequence of spaces Q_j and maps $q_j \colon X \to Q_j$ (with $j \geq n$) such that

(i) $(q_j)_* \colon \pi_k(X) \to \pi_k(Q_j)$ is an isomorphism for $k \leq n$,
(ii) $\pi_k Q_j = 0$ for $n < k \leq j$, and
(iii) For $j \geq n$, Q_{j+1} is obtained from Q_j by attaching $(j+2)$-dimensional cells.

We will then define P_n to be the colimit of the resulting diagram $X = Q_n \hookrightarrow Q_{n+1} \hookrightarrow Q_{n+1} \hookrightarrow \cdots$. First set $Q_n := X$ and define $q_n \colon X \to Q_n$ by the identity map id_X. Then q_n induces isomorphisms $\pi_k X \to \pi_k Q_n$ for $k \leq n$, so (i) holds, and (ii) and (iii) are vacuously true.

Next let Q_{n+1} be the space obtained by invoking Lemma 3.10.8.1 to glue $(m+2)$-cells to X to kill off each nontrivial homotopy class $[f] \in \pi_{n+1} X$, from which we obtain a map $q_{n+1} \colon X \to Q_{n+1}$. Since we added no cells of dimension $\leq n$, (i) again holds, and by construction $\pi_{n+1} Q_{n+1} = 0$ so (ii) holds, and finally (iii) holds again by construction. Continuing inductively, in this way, we obtain such maps $q_j \colon X \to Q_j$ for all $j \geq n$. Note that Lemma 3.10.8.1 also gives us inclusion maps $i_j \colon Q_j \hookrightarrow Q_{j+1}$ for all $j \geq n$.

Our construction yields the following cone.

$$\begin{tikzcd}
 & & X \arrow[dll, "q_1"'] \arrow[dl, "q_2"] \arrow[d, "q_3"] \arrow[dr, "q_4"] & & \\
Q_n = X \arrow[r, hook, "i_n"] & Q_{n+1} \arrow[r, hook, "i_{n+1}"] & Q_{n+2} \arrow[r, hook, "i_{n+2}"] & Q_{n+3} \arrow[r, hook] & \cdots
\end{tikzcd}$$

Thus, by setting

$$P_n(X) \coloneqq \lim \left(Q_n \xhookrightarrow{i_n} Q_{n+1} \xhookrightarrow{i_{n+1}} Q_{n+2} \xhookrightarrow{i_{n+2}} \cdots \right),$$

we obtain a unique map $\alpha_n \colon X \to P_n(X)$ compatible with both the above cone and the limit cone, and one checks that $\pi_j(P_n(X)) = \pi_j X$ for $j \leq n$ and $\pi_j(P_n(X)) = 0$ for $j \geq n+1$.

See [Str11, Problem 16.36] for a guide for the proof of the uniqueness statement. ☐

We now connect the Postnikov approximations into an inverse system called a *Postnikov tower*.

$$\begin{tikzcd}
 & \vdots \arrow[d, "p_{n+2}"'] & \\
F(p_{n+1}) = K(\pi_{n+1}X, n+1) \arrow[r, hook] & P_{n+1}(X) \arrow[d, "p_{n+1}"'] & \\
F(p_n) = K(\pi_n X, n) \arrow[r, hook] & P_n(X) \arrow[d, "p_n"'] & X \arrow[l, "\alpha_n"'] \arrow[ul, "\alpha_{n+1}"'] \arrow[ddl, "\alpha_1"] \\
 & \vdots \arrow[d, "p_2"'] & \\
 & P_1(X) &
\end{tikzcd}$$

Here the maps p_n are obtained by replacement of the the comparison map

$$P_n(X) = \operatorname{colim}(X \hookrightarrow Q_{n+1} \hookrightarrow \cdots) \to \operatorname{colim}(X \hookrightarrow Q_n \hookrightarrow Q_{n+1} \hookrightarrow \cdots) = P_{n-1}(X)$$

with a fibration.

Given how similar the construction of the Postnikov tower is to the construction of Eilenberg–MacLane spaces, it is no surprise that the (homotopy) fiber of each p_n is an Eilenberg–MacLane space $K(\pi_n X, n)$. Indeed, by the long exact sequence of homotopy groups induced by the fiber sequence $F(p_n) \hookrightarrow P_n(X) \xrightarrow{p_n} P_{n-1}(X)$, the exactness of

$$\cdots \longrightarrow \pi_{n+1}(\cancelto{0}{P_{n-1}(X))} \longrightarrow \pi_n(F(p_n)) \longrightarrow \cancelto{\pi_n X}{\pi_n(P_n(X))} \longrightarrow \cancelto{0}{\pi_n(P_{n-1}(X))} \longrightarrow \cdots$$

implies $\pi_n(F(p_n)) \cong \pi_n X$, and similarly the exactness of

$$\cdots \longrightarrow \pi_{k+1}(P_{n-1}(X)) \longrightarrow \pi_k(F(p_n)) \longrightarrow \pi_k(P_n(X)) \longrightarrow \pi_k(P_{n-1}(X)) \longrightarrow \cdots$$

implies $\pi_k(F(p_n)) = 0$ for all $k \neq n$ (because then the two terms to the right of $\pi_k(F(p_n))$ must be either both 0 (if $k > n$) or both $\pi_k X$ (if $k < n$) and the two terms to the left of

$\pi_k(F(p_n))$ must be either both 0 (if $k > n$) or both $\pi_{k+1}X$; in each of these cases, $\pi_k(F(p_n))$ must be 0 to preserve exactness).

3.10.9.2. *Whitehead towers.* Whitehead towers are "dual" to Postnikov towers in the sense that **TODO:**

3.10.9.3. *Maps into Eilenberg–MacLane spaces.* See [Str11, §17.6.1]

3.10.9.4. *Group cohomology as singular cohomology.* We define the nth *group cohomology* of a group G with coefficients in an abelian group π to be $H^n(G;\pi) \coloneqq H^n(K(G,1);\pi)$.

3.10.9.5. *Eilenberg–MacLane spaces represent cohomology.* Let π be an abelian group. By the UCT for cohomology, there is a SES

$$0 \longrightarrow \underset{\mathbb{Z}}{\overset{1}{\mathrm{Ext}}}(H_{n-1}K(\pi,n),\pi) \longrightarrow H^n(K(\pi,n);\pi) \xrightarrow{\mathrm{ev}} \underset{\mathbb{Z}}{\mathrm{Hom}}(H_nK(\pi,n),\pi) \longrightarrow 0. \quad (3.10.9.3)$$

As $\pi_qK(n,\pi) = 0$ for $i \leq n-2$, by Hurewicz's theorem $H_{n-1}K(\pi,n) \cong \pi_{n-1}K(\pi,n) = 0$, which is a free module (it has the empty basis). Thus $\mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}K(\pi,n),\pi) = 0$, so

$$H^n(K(\pi,n);\pi) \overset{(3.10.9.3)}{\cong} \underset{\mathbb{Z}}{\mathrm{Hom}}(H_nK(\pi,n),\pi) \cong \underset{\mathbb{Z}}{\mathrm{Hom}}(\pi_nK(\pi,n),\pi) = \underset{\mathbb{Z}}{\mathrm{Hom}}(\pi,\pi),$$

where we used Hurewicz's theorem again for the third isomorphism. We call the cohomology class $[[\alpha]] \in H^n(K(\pi,n);\pi)$ corresponding to the identity $\mathrm{id}_\pi\colon \pi \to \pi$ the *fundamental class.*

Theorem 3.10.9.4. *For $n \in \mathbb{Z}$ and $X \in \mathcal{T}_*$, there is a canonical isomorphism*

$$H^n(X;\pi) \cong [X, K(\pi,n)]_*,$$

natural in $X \in \mathcal{T}_*$, that sends a pointed homotopy class $[f] \in [X,K(\pi,n)]_*$ to the pullback of the fundamental class, i.e., $f^*[[\alpha]] = [[\alpha \circ f]] \in H^n(X;\pi)$.

Proof. See here. ☐

3.10.10. **Ω-spectra.** An Ω-*spectrum* is a sequence $\{(K_n,\sigma_n)\}_{n\geq 0}$ of pointed spaces K_n together with weak homotopy equivalences $\sigma_n\colon K_n \to \Omega K_{n+1}$ for all $n \geq 0$.

3.10.11. **Generalized (co)homology theories.** A *generalized reduced cohomology theory* (GRCT) consists of strictly homotopy-invariant functors $\{\widetilde{h}^n\colon (\mathcal{T}_*^{\mathrm{CW}})^{\mathrm{op}} \to \mathsf{Ab}\}_{n\geq 0}$ together with natural isomorphisms $\sigma_n\colon \widetilde{h}^n \Rightarrow \widetilde{h}^{n=1} \circ \Sigma$ satisfying

- (additivity) $\widetilde{h}^n\left(\bigvee_\alpha X_\alpha\right) \xrightarrow{\bigvee_\alpha i_\alpha} \coprod_\alpha \widetilde{h}^n(X_\alpha)$, and

- (exact sequence) for all pointed (without loss of generality, cellular) maps $f\colon X \to Y$, the inclusion $Y \hookrightarrow Cf$ of Y into the mapping cone induces an exact sequence $\widetilde{h}^n(Cf) \to \widetilde{h}^n(Y) \to \widetilde{h}^n(X)$. (In particular, this means the strict cofiber sequence $A \subseteq X \twoheadrightarrow X/A$ induces an exact sequence $\widetilde{h}^n(X/A) \to \widetilde{h}^n(A) \to \widetilde{h}^n(X)$.)

Theorem 3.10.11.1. *For an Ω-spectrum $\{(K_n, \sigma_n)\}_{n\geq 0}$, $\widetilde{h}^n := [-, K_n]_*$ defines a generalized reduced cohomology theory. Moreover, when $K_n = K(\pi, n)$ and $\sigma_n = \Sigma$ for all $n \geq 0$, then $\widetilde{h}^n := H^n(-;\pi)$.*

Proof. Obviously $[-, K_n]$ is homotopy-invariant. Define $\sigma_n\colon [-, K_n] \Rightarrow [\Sigma(-), K_{n+1}]$ at $X \in \mathcal{T}_*$ by

$$\widetilde{h}^n X = [X, K_n] \xrightarrow{(w_n)_*} [X, \Omega K_{n+1}] \xrightarrow{\Sigma\dashv\Omega} [\Sigma X, K_{n+1}] = \widetilde{h}^{n+1}(\Sigma X)$$

where $w_n\colon K_n \to \Omega K_{n+1}$ is the nth weak equivalence from the given Ω-spectrum. The first map is a bijection by the first clause of Whitehead's theorem. This bijection is natural in X because if $f \in (\mathcal{T}_*^{\mathrm{CW}})^{\mathrm{op}}(X' \to X)$, then the diagram

$$\begin{tikzcd}
{[X, K_n]} \arrow[r, "(w_n)_*"] \arrow[d, "f^*"'] & {[X, \Omega K_{n+1}]} \arrow[d, "f^*"] \\
{[X', K_n]} \arrow[r, "(w_n)_*"] & {[X', \Omega K_{n+1}]}
\end{tikzcd}$$

commutes, as $(w_n)_* f^*[g] = w_n \circ [g \circ f] = [w_n \circ g \circ f]$ equals $f^{(}(w_n)_*[g] = [w_n \circ g \circ f]$. Hence the σ_n's are natural bijections. And in fact they are group isomorphisms by considering the group structure on K_n for $n \geq 1$ (because the homotopy sets have group structure by concatenation of loops, which one can check is compatible with everything we have done here.) And additivity is clear, so we have proved the first claim.

It remains to prove that the Ω-spectra of Eilenberg–MacLane spaces gives ordinary singular cohomology. **TODO: Replace this paragraph by saying something like ”we have already proved this before” and then cite it, but only do this after you actually prove it two sections ago.**

For the second claim, consult, e.g., [May11]. □

3.11. (WIP) Spectra

3.12. Homotopy group actions

The next paragraph largely follows [Str11, §15.3].

For a group G, a *G-space* is a space $X \in \mathcal{T}_\circ$ equipped with a G-action. This gives maps $g\colon X \to X$, and we typically write $g \cdot x$ for $g(x)$. A map $f\colon X \to Y$of G-spaces is

G-equivariant if it commutes with the G-actions in the sense that $f(g \cdot x) = g \cdot f(x)$. When G is identified as a category with a single object, the category of G-shaped diagrams in $\mathcal{T}_\circ$ is just the category of G-spaces and G-equivariant maps. For a G-space X, we write X/G for the quotient space $X/\sim$ where $x \sim g \cdot x$ for all $x \in X$ and $g \in G$.

The rest of this section largely follows parts of [Str11, §8.8.2–3].

Fix a group G. Where $\underline{G}$ is the single-object $(*)$ category with $\operatorname{End}(*) = G$, the diagram category $\mathsf{Fun}(\underline{G} \to \mathcal{T}_\circ)$ is the category whose objects are the G-spaces in $\mathcal{T}_\circ$ and whose morphisms are the G-maps, i.e., G-equivariant maps. Indeed, $* \mapsto X$ picks out the underlying space, while G-maps, i.e., maps $f\colon X \to Y$ such that $f(g \cdot x) = g \cdot f(x)$ for all $x \in X$, make the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

commute, and hence are natural transformations between functors $* \to X$ and $* \to Y$.

Call x and x' *equivalent* if $G \cdot x = G \cdot y$; the quotient X/G of X by this relation is called the *orbit space*. Call $x \in X$ such that $G \cdot x = \{x\}$ a *fixed point* by the G-action, and we denote by X^G the set of such points. Observe the following for a G-action $\mathscr{A}\colon \underline{G} \to \mathcal{T}_\circ$ on X.

- $\operatorname{colim} \mathscr{A} = X/G$. Indeed, the diagram

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, two heads] & & X \arrow[dl, two heads] \\
& X/G &
\end{tikzcd}$$

  commutes because $g \cdot (G \cdot x) = G \cdot x$ for all g and x, and any $Q \in \mathcal{T}_\circ$ with the same property is factored through by $\operatorname{colim} \mathscr{A}$ uniquely, since if the diagram

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "p"'] & & X \arrow[dl, "q"] \\
& Q &
\end{tikzcd}$$

  commutes for all g, then $q(g \cdot x) = p(x)$ for all g and x, which says q is constant on orbits. Thus q descends to a unique map $X/G \to Q$ by the universal property of the quotient.
- $\lim \mathscr{A} = X^G$. Indeed, the diagram

$$\begin{tikzcd}
& X^G \arrow[dl, hook] \arrow[dr, hook] & \\
X \arrow[rr, "g"] & & X
\end{tikzcd}$$

  commutes for all $g \in G$ by definition of fixed points, and any $W \in \mathcal{T}_\circ$ with the same property factors uniquely through X^G, because for a commutative diagram

$$\begin{tikzcd} & X^G \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, "g"] & & X \end{tikzcd}$$

we have $g \cdot p(x) = q(x)$ for all g implies $g \cdot p = q$, i.e., $p(X) = X^G$. Thus there is a unique map $W \to X^G$ making the relevant diagram commute.

Next observe that the assignments $X \mapsto X^G$ and $X \mapsto X/G$ give functors $(-)^G\colon \mathsf{Fun}(\underline{G} \to \mathcal{T}_\circ) \to \mathcal{T}_\circ$ and $(-)/G\colon \mathsf{Fun}(\underline{G} \to \mathcal{T}_\circ) \to \mathcal{T}_\circ$ respectively. In fact, where $\Delta\colon \mathcal{T}_\circ \to \mathsf{Fun}(\underline{G} \to \mathcal{T}_\circ)$ assigns $X \in \mathcal{T}_\circ$ the trivial G-action, we have adjunctions $(-)/G \dashv \Delta \dashv (-)^G$.

Actions of groups on spaces can be "nasty and lead to pathological and uninformative fixed points and orbit spaces" [Str11, p. 218]. The solution is to consider fixed points and orbit spaces *up to homotopy*, which, respectively, we define to be holim and hocolim of the action $\mathcal{A}\colon \underline{G} \to \mathcal{T}_\circ$.

Unfortunately, for any nontrivial group G, the category $\underline{G}$ thought of as a diagram category is neither simple nor cosimple categories (i.e., neither $\underline{G}$ nor $\underline{G^{\mathrm{op}}}$ are simple), so our construction for fibrant and cofibrant replacements do not apply. However, there are some things that we can still say. It turns out that all G-CW-complexes *are* cofibrant; a *G-CW-complex* is a CW-complex X that is a G-space such that G *acts cellularly* on X, i.e., each $g \in G$ sends open cells to themselves (but not necessarily fixing the points within those cells!) and with the additional property that only the identity $1 \in G$ can send a cell to itself.

3.12.0.1. *A necessary condition for cofibrant G-spaces.* [Str11, §16.5] shows that for all groups G, there is a contractible free G-CW-complex $\mathrm{E}G$. Then [Str11, Problem 8.70] shows that for a G-CW-complex XS, the conditions that (i) there is a G-map $X \to \mathrm{E}G$ and (ii) X is a free G-CW-complex are equivalent. Then [Str11, Problem 8.71] shows that for a G-space X, if X is a cofibrant diagram, then G acts freely on X; this is shown by considering the following lifting problem.

$$\begin{tikzcd} & \mathrm{E}G \arrow[d] \\ X \arrow[ur, dashed] \arrow[r] & * \end{tikzcd}$$

3.12.0.2. *A sufficient condition for cofibrant G-spaces.* [Str11, Problem 8.72] shows that all free G-CW-complexes are cofibrant diagrams in $\mathcal{T}_\circ^G$.

Chapter 4

Towards Bott Periodicity

4.1. CLASSIFYING VECTOR BUNDLES

TODO: type up all the (massive amount of) notes from notebook that I have for this stuff...

4.1.1. **Pullback bundles.** Denote the set of isomorphism classes of real (resp. complex) n-dimensional vector bundles over B by $\mathsf{Vec}^n(B)$ (resp. $\mathsf{Vec}^n_{\mathbb{C}}(B)$, and let $\mathsf{Vec}(B)$ (resp. $\mathsf{Vec}_{\mathbb{C}}(B)$) denote the set of isomorphism classes of vector bundles over B of any dimension.

Proposition 4.1.1.1 ([Hat17, Proposition 1.5])**.** *Given a vector bundle $E \to B$and a map $f\colon A \to B$, there is a vector bundle $p'\colon E' \to A$ and a map $f'\colon E' \to E$taking the fiber of E' over each point $a \in A$ isomorphically onto the fiber of E over $f(a)$, and such a vector bundle $E'$is unique up to isomorphism.*

Thus any map $f\colon A \to B$ induces a set map $f^*\colon \mathsf{Vec}(B) \to \mathsf{Vec}(A)$. Often the bundle E' corresponding to E is denoted f^*E, and is called the *pullback* of E by f. One can show pullbacks satisfy (i) $(fg)^*E \cong g^*(f^*E)$, (ii) $\mathrm{id}_E^* \cong E$, (iii) $f^*(E_1 \oplus E_2) \cong f^*E_1 \oplus f^*E_2$, and (iv) $f^*(E_1 \otimes E_2) \cong f^*E_1 \otimes f^*E_2$.

The same proof for the following theorem works for fiber bundles, not just vector bundles.

Theorem 4.1.1.2 ([Hat17, Theorem 1.6])**.** *For a vector bundle $p\colon E \to B$ over a paracompact space B and homotopic maps $f_0, f_1\colon A \to B$, the induced bundles $f_0^*(E)$ and $f_1^*(E)$ are isomorphic.*

Corollary 4.1.1.3 ([Hat17, Corollary 1.8])**.** *A homotopy equivalence $f\colon A \to B$ of paracompact spaces induces a bijection $f^*\colon \mathsf{Vec}^n(B) \to \mathsf{Vec}^n(A)$. In particular, every vector bundle over a contractible paracompact space is trivial.*

4.1.2. **Universal principal bundles.** **TODO: Redo this whole section using §2 of Danny Calegari's notes here.** Here we record the useful result that G-bundles over a paracompact space are classified by certain homotopy classes.

Fix a topological group G. One first constructs simplicial topological spaces $E_*G, B_*G \in \mathsf{sTop}$ and a simplicial map $p_*\colon E_*G \to B_*G$ such that G acts freely on each $E_n(G)$ on the right and $B_n(G) \cong E_n(G)/G$. One then applies the geometric realization functor $|-| : \mathsf{sTop} \to \mathsf{Top}$ to obtain a map $p\colon BG \to EG$ which turns out to be a very nice bundle, as the following theorem shows.

Theorem 4.1.2.1. *If a topological group G is well-pointed with basepoint the identity, then there is a contractible space EG on which G acts freely and a principal G-bundle $G \hookrightarrow EG \to BG \cong EG/G$ that is universal in the sense that there is a natural bijection*

$$\begin{aligned} \left\{\begin{matrix} \textit{principal } G\textit{-bundles over} \\ \textit{a paracompact space } X \end{matrix}\right\} \quad &\cong \quad [X, BG], \\ f^*(EG) \quad &\leftarrow\!\shortmid \quad f \end{aligned}$$

We call BG the classifying space *of G, and this has the following additional properties: (i) For a topological group H satisfying the same hypotheses as G, $B(G \times H) \cong BG \times BH$. (ii) BG has multiplication $B(G \times G) \xrightarrow{\cong} BG \times BG \to BG$. (iii) When G is abelian, BG and EG are abelian. (iv) BG is unique up to homotopy equivalence. (v) $\pi_{q+1}(BG) \cong \pi_q(G)$ for all $q \geq 0$.*

Construction 4.1.2.2. Here we construct the classifying and universal spaces for a topological group G. Recall the *simplex category* Δ consists of finite ordered sets and order-preserving maps between them. The isomorphism classes of Δ then consist of the set $\{[n] \mid n \in \mathbb{Z}_{\geq 0}\}$ and one can show it is spanned by collections of maps, the *face maps* $\{d_i\colon [i] \to [i-1] \mid i \in \mathbb{Z}_{\geq 0}\}$ obtained by omitting the ith vertex and the *degeneracy maps* $\{s_i\colon [i] \to [i+1] \mid i \in \mathbb{Z}_{\geq 0}\}$ obtained by repeating the ith vertex. A *simplicial space* is a functor $\Delta^{\mathrm{op}} \to \mathsf{Top}$, or equivalently an assignment of a topological space X_n for each $n \in \mathbb{Z}_{\geq 0}$ and of maps $d_i\colon X_{i-1} \to X_i$ and $s_i\colon X_{n+1} \to X_n$ satisfying the simplicial identities.

Now, for each $n \geq 0$, define simplicial spaces $E_\bullet(G), G_\bullet(G)\colon \Delta \to \mathsf{Top}$ by $E_n(G) \coloneqq G^{\times(n+1)}$, $B_n(G) \coloneqq G^{\times n}$, let $p_n\colon E_n(G) \to B_n(G)$ be the projection onto the first n coordinates, and define face maps $d_i\colon E_{n+1}(G) \to E_{n-1}(G)$, and degeneracy maps $s_i\colon E_{n-1}(G) \to E_n(G)$ respectively by

$$d_i(g_1, \ldots, g_{n+1}) \coloneqq \begin{cases} (g_2, \ldots, g_{n+1}) & \text{if } i = 0, \\ (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}) & \text{if } i \in \{1, \ldots, n\} \end{cases}$$

and $s_i(g_1, \ldots, g_{n+1}) \coloneqq (g_1, \ldots, g_{i-1}, 1_G, g_i, \ldots, g_n)$. These maps are defied similarly for $B_\bullet(G)$, but its version of d_n is $d_n(g_1, \ldots, g_n) \coloneqq (g_1, \ldots, g_{n-1})$.

Notice G acts on $E_n(G)$ freely on the right by $(g_1, \ldots, g_{n+1}) \cdot g \coloneqq (g_1, \ldots, g_n, g_{n+1}g)$. This makes $E_\bullet(G)$ a *simplicial G-space*, i.e., the G-action commutes with the face and degeneracy maps. Clearly $B_n(G) = E_n(G)/G$, since $B_n(G) = G^n \subseteq G^{n+1} = E_n(G)$ is precisely what is fixed by the G-action as G acts transitively on itself in the last coordinate.

Now apply the geometric realization functor $|-| : \mathsf{sTop} \to \mathsf{Top}$ to obtain $BG := |B_\bullet(G)|$, $EG := |E_\bullet(G)|$, so that $p\colon BG \to EG$ is $|p_\bullet|$. Moreover, EG inherits a free right G-action, $BG$is the orbit space EG/G, and EG is contractible.

We call BG the *classifying space* of G and we call $p\colon EG \to BG$ the *universal bundle*. If $(G, 1_G)$ is well-pointed, then p is indeed a fiber bundle over BG with fiber G. By the long exact sequence of homotopy groups, we thus obtain $\pi_{q+1}(BG) \cong \pi_q(G)$ for all $q \geq 0$. We have homeomorphisms $B(G \times H) \cong BG \times BH$, so B is a product preserving functor $\mathsf{TopGrp} \to \mathsf{TopGrp}$. If G is abelian, then so is BG and EG. In fact, multiplication on BG is the composite $BG \times BG \overset{\cong}{\to} B(G \times G) \to BG$. Lastly, and most importantly, we have a bijection

$$\begin{aligned} \left\{\begin{matrix}\text{principal } G\text{-bundles over} \\ \text{a paracompact space } X\end{matrix}\right\} &\quad \xrightarrow{\cong} \quad [X, BG], \\ f^*(EG) &\quad \longleftarrow \quad f \end{aligned}$$

We now present some more useful facts about classifying spaces.

The *delooping* of a topological space X is a topological space BX such that $X \simeq \Omega BA$. As the notation suggests, the classifying space of a space is its delooping.

Proposition 4.1.2.3. *For a topological space X,*
- $X \simeq \Omega(BX)$,
- *For a discrete group G, $H^k(G) \cong H^k(BG)$, where the left and right sides are group and singular cohomology respectively.*

4.1.3. **Construction of Eilenberg–MacLane spaces for abelian groups.** Classifying spaces allow us to construct Eilenberg–MacLane spaces for abelian groups. Indeed,since the inclusion $G \hookrightarrow EG$ as the fiber of p over the basepoint (which is the unique point in $B_0(G) = G^{\times 0} = \{1_G\}$) are group homomorphisms, we can iterate the process by setting $B^nG := B(B^{n-1}G)$ for $n \geq 2$, where $B^1G := BG$. We then set $K(G, n) := B^nG$ for each $n \geq 0$. Then

$$\pi_q(K(G,n)) = \pi_{q-1}(K(G,n-1)) = \cdots = \pi_{q-n}(K(G,0)) = \begin{cases} G & \text{if } q = n, \\ 0 & \text{if } q \neq n. \end{cases}$$

4.2. CHARACTERISTIC CLASSES

4.2.1. **Stiefel–Whitney classes.** See here.

4.3. K-THEORY

TODO: type up from notes

4.4. BOTT PERIODICITY THEOREM

TODO: type up from notes

Appendix A

Results from Homological Algebra

A.0.1. **Basic homological algebra.**

Lemma A.0.1.1 (Zig-zag lemma)**.** *For a SES of chain complexes* $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$*, there is the following LES of homology.*

$$\cdots \longrightarrow H_{q+1}(C_\bullet) \longrightarrow H_q(A_\bullet) \xrightarrow{f_*} H_q(B_\bullet) \xrightarrow{g_*} H_q(C_\bullet) \xrightarrow{\partial} H_{q-1}(A_\bullet) \longrightarrow \cdots$$

Lemma A.0.1.2. *Suppose the following diagram of abelian groups and group homomorphisms is commutative with exact rows.*

$$\begin{tikzcd}
\cdots \arrow[r] & C_{q+1} \arrow[r, "\partial_{q+1}"] \arrow[d, "c_{q+1}", "\cong"'] & A_q \arrow[r, "f_q"] \arrow[d, "a_q"] & B_q \arrow[r, "g_q"] \arrow[d, "b_q"] & C_q \arrow[r, "\partial_q"] \arrow[d, "c_q", "\cong"'] & A_{q-1} \arrow[r] \arrow[d, "a_{q-1}"] & \cdots \\
\cdots \arrow[r] & C'_{q+1} \arrow[r, "\partial'_{q+1}"] & A'_q \arrow[r, "f'_q"] & B'_q \arrow[r, "g'_q"] & C'_q \arrow[r, "\partial_q"] & A'_{q-1} \arrow[r] & \cdots
\end{tikzcd}$$

Then there is a LES of Mayer–Vietoris type*, i.e., a LES of the following form.*

$$\cdots \longrightarrow A_q \xrightarrow{\alpha_q} A'_q \oplus B_q \xrightarrow{\beta_q} B'_q \xrightarrow{\gamma_q} A_{q-1} \xrightarrow{\alpha_{q-1}} A'_{q-1} \oplus B_{q-1} \xrightarrow{\beta_{q-1}} B'_{q-1} \longrightarrow \cdots$$

In the proof of the above lemma, $\alpha_q \coloneqq a_q \oplus f_q$, $\beta_q \coloneqq b_q - f'_q$, and $\gamma_q \coloneqq bd_q \circ c_q^{-1} \circ g'_q$ and the LES is the following sequence in red.

$$\begin{tikzcd}
\cdots \arrow[r] & A_q \arrow[rr, "f_q"] \arrow[dr, "\alpha_q"] \arrow[dd, "a_q"] & & B_q \arrow[r, "g_q"] \arrow[dl, hook'] \arrow[dd, "b_q"] & C_q \arrow[r, "\partial_q"] \arrow[dd, "c_q"] & A_{q-1} \arrow[r] \arrow[dd, "a_{q-1}"] \arrow[dr] & \cdots \\
\cdots \arrow[ur] & & A'_q \oplus B_q \arrow[dr, "\beta_q"] & & & & \cdots \\
\cdots \arrow[r] & A'_q \arrow[rr, "f'_q"] \arrow[ur, hook] & & B'_q \arrow[r, "g'_q"] \arrow[uurr, "\gamma_q"] & C'_q \arrow[r, "\partial'_q"] & A'_{q-1} \arrow[r] & \cdots
\end{tikzcd}$$

Lemma A.0.1.3 (LES from a triple)**.** *For a topological triple* (X, V, A)*, there is a LES of* *homology groups of the following form.*

$$\cdots \longrightarrow H_k(V,A) \longrightarrow H_K(X,A) \longrightarrow H_k(X,V) \xrightarrow{\partial} H_{k-1}(V,A) \longrightarrow \cdots$$

Proof. Let $j : V \to X$ be the inclusion map. The induced homomorphism $j_* : C_q(V) \hookrightarrow C_q(X)$ descends to an injective map $\overline{j}_* : C_q(V)/C_q(A) \hookrightarrow C_q(X)/C_q(A)$ via the quotient map. On the other hand, the map sending chain residues $[c]_A$ in $C_q(X)/C_q(A)$ to $[c]_V$ in $C_q(X)/C_q(V)$ is surjective since $C_q(A)$ can be identified with a $\mathbb{Z}$-submodule of $C_q(V)$ (via i_*, where $i : A \hookrightarrow V$ is the inclusion map), so this map is also surjective. Hence there are maps making $C_q(V)/C_q(A) \hookrightarrow C_q(X)/C_q(A) \twoheadrightarrow C_q(X)/C_q(V)$ a SES. Thus we get a SES of complexes $0 \to C_\bullet(V,A) \to C_\bullet(X,A) \to C_\bullet(X,V) \to 0$, which is isomorphic to $0 \to C_\bullet(V,A)(\cong C_\bullet(V)/C_\bullet(A) \to C_\bullet(X,A)(\cong C_\bullet(X)/C_\bullet(A)) \to C_\bullet(X,V)(\cong C_\bullet(X)/C_\bullet(A)) \to 0$, so this gives the desired LES. □

Theorem A.0.1.4 (Five lemma). *If in the following commutative diagram in an abelian category the rows are exact, then h_3 is an isomorphism.*

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, two heads, "h_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2", "\cong"'] & A_3 \arrow[r, "f_3"] \arrow[d, "h_3"] & A_4 \arrow[r, "f_4"] \arrow[d, "h_4", "\cong"'] & A_5 \arrow[d, tail, "h_5"] \\
B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & B_3 \arrow[r, "g_3"] & B_4 \arrow[r, "g_4"] & B_5
\end{tikzcd}$$

A.1. Spectral sequences

Here we follow the beginning of [HP12]. We work with abelian groups but this section can be done in any abelian category. Other good resources for spectral sequences are here, here, here, here, here, and here.

A.1.1. **Bigraded abelian groups.** A *bigraded abelian group* is a collection of abelian groups $A = A_{\bullet,\bullet} = \{A_{p,q}\}_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}$. A *bigraded map* $f\colon A \to B$ of *bidegree* (a,b) is a family of group homomorphisms $f = \{f_{p,q}\colon A_{p,q} \to B_{p+a,q+b}\}_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}$. Write $\ker f \coloneqq \{\ker f_{p,q}\} \subseteq \{A_{p,q}\}$ and $\operatorname{im} f \coloneqq \{\operatorname{im} f_{p,q}\} \subseteq \{B_{p,q}\}$.

A.1.2. **Differential bigraded abelian groups.** A *differential bigraded abelian group* is a pair (E,d) where E is a bigraded abelian group and $d\colon E \to E$, called the *differential*, such that $d \circ d = 0$.

The *homology* $H_{\bullet,\bullet}(E,d)$ of a differential bigraded abelian group (E,d) is $H_{p,q}(E,d) \coloneqq \ker d_{p,q}/\operatorname{im} d_{p-a,q-b}$ where (a,b) is the bidegree of the differential d.

A.1.3. **Spectral sequences.** A *spectral sequence* of *homological type* is a collection of bigraded groups $\{(E^r_{\bullet,\bullet}, d^r)\}_{r\in\mathbb{Z}_{\geq 1}}$ where $E^{r+1}_{p,q} = H_{p,q}(E^r_{\bullet,\bullet}, d^r)$ and the differentials d^r have bidegree $(-r, r-1)$. The bigraded abelian group E^r is called the *E^r-page* or the *rth page* of the spectral sequence.

A.1.4. **The limit page of a spectral sequence.** To motivate the notion of limit pages, it is instructive to describe a spectral sequence $\{(E^r_{\bullet,\bullet}, d^r)\}$ in terms of subgroups of $E^1_{\bullet,\bullet}$.

First set $Z^1 \coloneqq \ker d^1$ and $B^1 \coloneqq \operatorname{im} d^1$. Then $B^1 \subseteq Z^1 \subseteq E^1$ and $E^2 = Z^1/B^1$ by definition.

Next, since $\ker d^2$ and $\operatorname{im} d^2$ are subgroups of the quotient group $E_2 = Z^1/B^1$, by the correspondence theorem there are subgroups Z^2 and B^2 of E^2 containing B^1 for which $\ker d^2 = Z^2/B^1$ and $\operatorname{im} d^2 = B^2/B^1$. Then $B^1 \subseteq B^2 \subseteq Z^2 \subseteq Z^1 \subseteq E^1$ and $E^3 = \ker d^3/\operatorname{im} d^3 = (Z^2/B^1)/(B^2/B^1) = Z^2/B^2$.

By iterating this process, we present the spectral sequence as the following infinite sequence of subgroups of E^1.

$$B^1 \subseteq \cdots \subseteq B^n \subseteq \cdots \subseteq Z^n \subseteq \cdots \subseteq Z^1 \subseteq E^1,$$

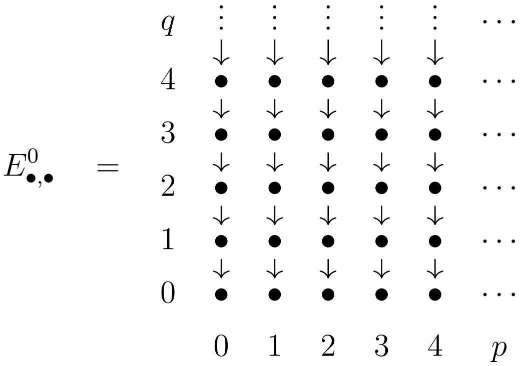
from which we recover E^n by the rule $E^{n+1} = Z^n/B^n$. Setting $Z^\infty_{\bullet,\bullet} \coloneqq \bigcap_{r=1}^\infty Z^r_{\bullet,\bullet}$ and $B^\infty_{\bullet,\bullet} \coloneqq \bigcap_{r=1}^\infty B^r_{\bullet,\bullet}$, the *limit page* of the spectral sequence $\{(E^r_{\bullet,\bullet}, d^r)\}_{r\in\mathbb{Z}_{\geq 1}}$ is $E^\infty_{\bullet,\bullet} \coloneqq Z^\infty_{\bullet,\bullet}/B^\infty_{\bullet,\bullet}$.

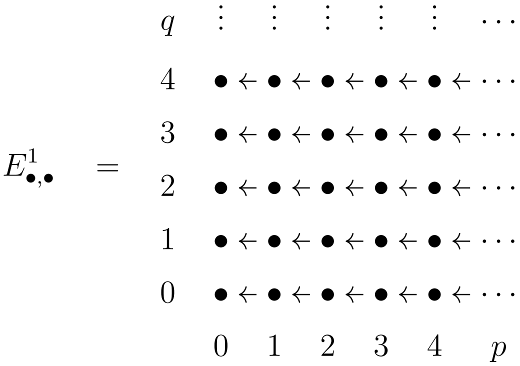
Exercise A.1.4.1 ([HP12, Lemma 1.1.5])**.**

(i) If $E^{r+1} = E^r$ for all $r \geq s$, then $E^s = E^\infty$.
(ii) $E^{r+1} = E^r$ if and only if $Z^{r+1} = Z^r$ and $B^{r+1} = B^r$.

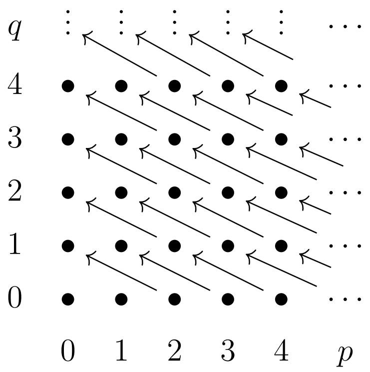
A *first-quadrant spectral sequence* is a spectral sequence $\left\{(E^r_{\bullet,\bullet}, d^r)\right\}$ with $E^r_{p,q} = 0$ when $p < 0$ or $q < 0$. The following corollary follows from the above discussion.

Corollary A.1.4.2. *For a first-quadrant homological spectral sequence, $E^r_{p,q} = E^\infty_{p,q}$ whenever $\max\{p, q+1\} < r < \infty$.*

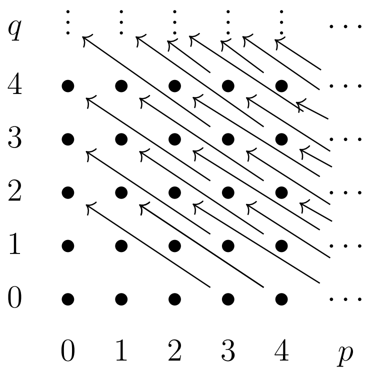




$E^2_{\bullet,\bullet} =$



$E^3_{\bullet,\bullet} =$



A.1.5. **Spectral sequences of cohomological type.** Arguably more common are spectral sequences of *cohomological type*, for which the (p,q)-term of the E^r-page is denoted $E_r^{p,q}$ and the boundary maps d_r are of bidegree $(r, -(r-1))$, i.e., a multiple of $(-1,-1)$ of the bidegree of those of homological type.

In the above diagrams showing the E_0-, E_1-, E_2-, and E_3-pages of a homological first-quadrant spectral sequence, just swap the source and target of each arrow to get the corresponding page for a cohomological first-quadrant spectral sequence.

A.1.6. **Convergence of spectral sequences.** A *filtration* $F_\bullet$ on an abelian group A is a family of subgroups $\{F_pA\}_{p\in\mathbb{Z}}$ of A such that $\cdots \subseteq F_{p-1}A \subseteq F_pA \subseteq F_{p+1}A \subseteq \cdots$. Each filtration F of A determines an *associated graded abelian group* $E^0_\bullet$ given by $\left\{E^0_p(A) = F_pA/F_{p-1}A\right\}$. If A itself is graded, then the filtration is assumed to preserve the grading in the sense that $F_pA_n \cap A_n \subseteq F_{p+1}A_n \cap A_n$ for all p and n. A filtered graded abelian group $F_\bullet A_\bullet$ is called *bounded* if there are finitely many nonzero filtration levels at any degree d, i.e., if for each d there are s and t such that $\{0\} = F_sC_d \subseteq F_{s+1}C_d \subseteq \cdots \subseteq F_{t-1}C_d \subseteq F_tC_d = C_d$.

The *associated graded abelian group* of a filtered graded abelian group $F_\bullet A_\bullet$ is $E^0_{p,q}(F_\bullet A_\bullet) \coloneqq F_pA_{p+q}/F_{p-1}A_{p+q}$.

A spectral sequence $\left\{(E^r_{\bullet,\bullet}, d^r)\right\}$ is said to *converge* to a graded abelian group $A_\bullet$ if there is a filtration $F_\bullet A_\bullet$ together with isomorphisms $E^\infty_{p,q} \cong E^0_{p,q}(F_\bullet A_\bullet)$. We write

$$E^1_{p,q} \Rightarrow A_{p+q}$$

to mean the spectral sequences converges to $A_\bullet$.

The *degree* of a term $E^r_{p,q}$ is $d \coloneqq p + q$.

A.1.7. **Example: Spectral sequence of a filtered complex.** A *filtered complex* is a pair $(F_\bullet C_\bullet, \partial)$ where $(C_\bullet, d)$ is a complex and $F_\bullet C_\bullet$ is a filtered graded abelian group such that

$\partial(F_pC_n) \subseteq F_pC_{n-1}$ for all p and n.

Note that the filtration induces a filtration $\{F_pH_n(C_\bullet)\}$ on $H_n(C_\bullet)$ given by the map induced on homology by the inclusion $F_pC_\bullet \hookrightarrow C_\bullet$.

Theorem A.1.7.1 ([HP12, Theorem 1.2.2])**.** *Let $(F_\bullet C_\bullet, \partial)$ be a filtered complex.*

(i) *There is a spectral sequence $\{(E^r_{\bullet,\bullet}, d^r)\}_{r\geq 1}$ with E^1-page $E^1_{p,q} = H_{p+q}(F_pC_\bullet/F_{p-1}C_\bullet)$ and maps $d^1_{p,q}\colon E^r_{p,q} \to E^r_{p-1,q}$ given by $[z + F_{p-1}C_\bullet] \mapsto [\partial z + F_{p-2}C_\bullet]$ where $z \in F_pC_\bullet$ satisfies $\partial z \in F_{p-1}C_\bullet$.*

(ii) *If $F_\bullet C_\bullet$ is bounded, then $E^1_{p,q} \Rightarrow H_{p+q}(C_\bullet)$, i.e.,*

$$E^\infty_{p,q} \cong \frac{F_pH_{p+q}(C_\bullet, \partial)}{F_{p-1}H_{p+q}(C_\bullet, \partial)}.$$

Thus $E^\infty_{p,q}$ determines $F_pH_{p+q}(C_\bullet, \partial)$ up to a group extension.

A.1.8. **Example: Spectral sequence of a double complex.** A *double complex* is a triplet $(C_{\bullet,\bullet}, \partial', \partial'')$ where $C_{\bullet,\bullet}$ is a bigraded abelian group and $\partial', \partial''\colon C_{\bullet,\bullet} \to C_{\bullet,\bullet}$, called the *horizontal* and *vertical boundary*, are bigraded maps, of bidegree $(-1, 0)$ and $(0, -1)$ respectively, satisfying the conditions $\partial' \circ \partial' = \partial'' \circ \partial'' = 0$ and ∂' and ∂'' anticommute, i.e., $\partial'_{p,q-1}\partial''_{p,q} = -\partial''_{p-1,q}\partial'_{p,q}$ for all p and q. Each double complex $(C_{\bullet,\bullet}, \partial', \partial'')$ gives a complex $(\mathrm{tot}(C)_\bullet, \partial)$, called the *total complex*, with terms $\mathrm{tot}(C)_d = \bigoplus_{p+q=d} C_{p,q}$ and boundary maps $\partial := \partial' + \partial''$.

For a double complex $(C, \partial', \partial'')$, the *homology of the rows* is the bigraded abelian group $H'_{\bullet,\bullet}(C)$ whose terms are $H'_{p,q}(C) := \ker \partial'_{p,q}/\mathrm{im}\,\partial'_{p+1,q}$, and the *homology of the columns* is the bigraded abelian group $H''_{\bullet,\bullet}(C)$ whose terms are $H''_{p,q}(C) := \ker \partial'_{p,q}/\mathrm{im}\,\partial'_{p,q+1}$.

Theorem A.1.8.1 ([HP12, Theorem 1.3.5])**.** *Let $(C_{\bullet,\bullet}, \partial', \partial'')$ be a double complex.*

(i) *There is a spectral sequence $\{({}_\rightarrow E^r_{\bullet,\bullet}, d^r)\}_{r\geq 1}$ with ${}_\rightarrow E^1_{p,q} = H''_{p,q}(C)$ and ${}_\rightarrow E_{p,q} \cong H'_pH''_q(C)$.*

(ii) *There is a spectral sequence $\{({}_\uparrow E^r_{\bullet,\bullet}, d^r)\}_{r\geq 1}$ with ${}_\uparrow E^1_{p,q} = H'_{p,q}(C)$ and ${}_\uparrow E_{p,q} \cong H''_pH'_q(C)$.*

(iii) *If $C_{p,q} = 0$ when $p < 0$ or $q < 0$, then both of the induced spectral sequences are first-quadrant and ${}_\rightarrow E^1_{p,q} \Rightarrow H_{p+q}(\mathrm{tot}(C)_\bullet, \partial)$ and ${}_\uparrow E^1_{p,q} \Rightarrow H_{p+q}(\mathrm{tot}(C)_\bullet, \partial)$.*

A.1.8.1. *Example: Snake lemma.* We can use Theorem A.1.8.1 to show that a diagram

$$\begin{tikzcd}
 & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & D \arrow[r] & E \arrow[r] & F &
\end{tikzcd}$$

with exact rows induces an exact sequence $\ker\alpha \to \ker\beta \to \ker\gamma \to \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma$. First reflect the diagram across the vertical axis.

We first consider the spectral sequence $\left\{({}_{\rightarrow}E^r_{\bullet,\bullet}, d^r)\right\}_{r\geq 1}$. The ${}_{\rightarrow}E^0_{\bullet,\bullet}$-page is the following.

$$\begin{tikzcd}
0 & C \arrow[l] \arrow[d, "\gamma"'] & B \arrow[l] \arrow[d, "\beta"'] & A \arrow[l] \arrow[d, "\alpha"'] \\
 & F & E \arrow[l] & D \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Taking homology to get the ${}_{\rightarrow}E^1_{\bullet,\bullet}$-page, by exactness of the rows we obtain the zero double complex. Thus the spectral sequence has already stabilized at the $E^1_{\bullet,\bullet}$-page, so $E^\infty_{p,q} = 0$ for all p and q.

Next we instead consider the spectral sequence $\left\{({}_{\uparrow}E^r_{\bullet,\bullet}, d^r)\right\}_{r\geq 1}$. Its ${}_{\uparrow}E^0_{\bullet,\bullet}$-page is the following.

$${}_{\uparrow}E^0_{\bullet,\bullet} \quad := \quad \begin{tikzcd}
0 & C \arrow[l] \arrow[d, "\gamma"'] & B \arrow[l] \arrow[d, "\beta"'] & A \arrow[l] \arrow[d, "\alpha"'] \\
 & F & E \arrow[l] & D \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Here we are indexing so that, for example, $C = {}_{\uparrow}E^0_{0,1}$ and $D = {}_{\uparrow}E^0_{0,2}$. By taking homology we get the terms of the $E^1_{\bullet,\bullet}$-page as follows.

$${}_{\uparrow}E^1_{\bullet,\bullet} \quad := \quad \begin{tikzcd}
0 & \ker\gamma \arrow[l] \arrow[d, "\gamma"'] & \ker\beta \arrow[l] \arrow[d, "\beta"'] & \ker\alpha \arrow[l] \arrow[d, "\alpha"'] \\
 & \operatorname{coker}\gamma & \operatorname{coker}\beta \arrow[l] & \operatorname{coker}\alpha \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Here we are indexing so that, for example, $\ker\gamma = {}_{\uparrow}E^1_{0,1}$ and $\operatorname{coker}\alpha = {}_{\uparrow}E^1_{0,2}$. Taking homology again to get the ${}_{\uparrow}E^2_{\bullet,\bullet}$-page, we obtain the following.

$${}_{\uparrow}E^2_{\bullet,\bullet} \quad := \quad \begin{tikzcd}
0 & ?? \arrow[l] \arrow[d, "\gamma"'] & ? \arrow[l] \arrow[d, "\beta"'] & ? \arrow[l] \arrow[d, "\alpha"'] \\
 & ? & ? \arrow[l] & ?? \arrow[l] \arrow[ull, "\delta" description] & 0 \arrow[l]
\end{tikzcd}$$

Here, as usual, we are omitting those differentials that are zero. Notice that all terms marked '?' stabilize at the ${}_{\uparrow}E^2_{\bullet,\bullet}$-page, so ${}_{\uparrow}E^2_{p,q} = {}_{\uparrow}E^\infty_{p,q}$, and that all the terms marked '??' stabilize at the ${}_{\uparrow}E^3_{\bullet,\bullet}$-page, so ${}_{\uparrow}E^3_{p,q} = {}_{\uparrow}E^\infty_{p,q}$. But by Theorem A.1.8.1(iii), so ${}_{\uparrow}E^\infty_{\bullet,\bullet} = {}_{\rightarrow}E^\infty_{\bullet,\bullet} = 0$, so the terms marked by '?' are 0 and the homology of the complex $0 \to ?? \xrightarrow{\delta} ?? \to 0$ is zero, which says $0 \to ?? \xrightarrow{\delta} ?? \to 0$ is exact. Thus δ is an isomorphism, so we can set $\partial := \delta^{-1}$ to obtain a sequence

$$\ker\alpha \to \ker\beta \to \ker\gamma \xrightarrow{\partial} \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma.$$

The fact that the terms marked '?' are zero means $H_1(\ker\alpha \to \ker\beta \to \ker\gamma) = 0$ and $H_1(\operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma) = 0$, which gives exactness at $\ker\beta$ and at $\operatorname{coker}\beta$ respectively. It remains to show exactness of the above sequence at $\ker\beta$ and $\operatorname{coker}\alpha$, or equivalently that $\ker \partial = \operatorname{im}(\ker \beta \to \ker \gamma)$ and $\ker(\operatorname{coker} \alpha \to \operatorname{coker} \beta) = \operatorname{im} \partial$ respectively.

Since the terms marked '??' are the same and the one on the bottom-right and on the top-left are $\ker(\operatorname{coker} \alpha \to \operatorname{coker} \beta)/0 = \ker(\operatorname{coker} \alpha \to \operatorname{coker} \beta)$ and $\ker \gamma / \operatorname{im}(\ker \beta \to \ker \gamma) = \operatorname{coker}(\ker \beta \to \ker \gamma)$, we find

$$\ker(\operatorname{coker} \alpha \to \operatorname{coker} \beta) = \operatorname{coker}(\ker \beta \to \ker \gamma).$$

But this means $\ker \partial = \operatorname{im}(\ker \beta \to \ker \gamma) = \ker(\operatorname{coker} \alpha \to \operatorname{coker} \beta) = \operatorname{im} \partial$, so we are done by our previous remarks.

A.1.8.2. *Exercise: Prove the 5-lemma with spectral sequences.*

A.1.9. **(Co)homology with coefficients twisted by systems of local coefficients.** For a more concrete approach, see [McC01, §5.3, pp. 163–6]. For a more categorical approach, see here. See **TODO: link in goodnotes** for a definition of local coefficient systems as a locally constant sheaf, which is equivalent to the previous two approaches for suitably nice spaces. Also see [Hat02, §3.H, p. 327] for an approach using universal covering spaces.

A.1.10. **Leray–Serre spectral sequence.** Here we record some main points from [McC01, Chapter 5]. Other helpful sources include [Str11, §31.1] and here. Note that the Leray–Serre spectral sequence is also called the Serre spectral sequence.

Theorem A.1.10.1 (Leray–Serre homological spectral sequence [McC01, Theorem 5.1, adapted])**.** *For an abelian group M and a fibration $F \hookrightarrow E \xrightarrow{\pi} B$ where B is path-connected and F is connected, there exists a first-quadrant spectral sequence $\{(E^r_{\bullet,\bullet}, \partial^r)\}$ with*

$$E^2_{p,q} = H_p(B; \mathcal{H}_q(F; G)) \quad \Rightarrow \quad H_{p+q}(E; G).$$

Moreover, this spectral sequence is natural with respect to fiber-preserving maps of fibrations.

In particular, if also B is path-connected and simply connected (i.e., 1-connected), then the system of local coefficients $\mathcal{H}_q(F; G)$ above can be replaced with ordinary homology $H_q(F; G)$.

The cohomological version of the Serre spectral sequence is more powerful, since it says a lot about the cup product for and thus the cohomology ring of a space.

Theorem A.1.10.2 (Leray–Serre cohomological spectral sequence [McC01, Theorem 5.2, adapted])**.** *For a commutative ring R and a fibration $F \hookrightarrow E \xrightarrow{\pi} B$ where B is path-connected and F is connected, there is a first-quadrant spectral sequence* of algebras *$\{(E_r^{\bullet,\bullet}, \partial_r)\}$ with*

$$E_2^{p,q} = H^p(B; \mathcal{H}^q(F; R)) \quad \Rightarrow \quad H^{p+q}(E; R)$$

Moreover, this spectral sequence is natural with respect to fiber-preserving maps of fibrations.

Furthermore, the cup product $\smile$ on cohomology with local coefficients and the product $\cdot_2$ on the bigraded algebra $E_2^{\bullet,\bullet}$ are related by $u \cdot_2 v = (-1)^{p'q} u \smile v$ when $u \in E_2^{p,q}$ and $v \in E_2^{p',q'}$.

In particular, if also B is path-connected and simply connected (i.e., 1-connected), then the system of local coefficients $\mathcal{H}^q(F;G)$ above can be replaced with ordinary cohomology $H^q(F;G)$.

A.1.10.1. *Exercise: Computing a cohomology ring with spectral sequences.* Using the fibration $S^1 \to S^{2n+1} \to \mathbb{C}\mathrm{P}^n$, we will derive the cohomology ring $H^\bullet(\mathbb{C}\mathrm{P}^n)$ with the Serre spectral sequence.

We will write out the E_2-page using the rule $E_r^{p,q} = H^p(\mathbb{C}\mathbb{P}^n; H^q(S^1))$. Since $H^q(S^1) = \delta_{q=1,2}\mathbb{Z}$, the terms $E_2^{p,q}$ are zero for $q \geq 2$ and $H^q(\mathbb{C}\mathbb{P}^1;\mathbb{Z}) = H^q(\mathbb{C}\mathbb{P}^1)$ otherwise. Thus we start with the following for the E_2-page.

$$\begin{tikzcd}
q & \vdots & \vdots & \vdots & \vdots & \vdots & \\
2 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & H^0(\mathbb{C}\mathrm{P}^n) \arrow[drr] & H^1(\mathbb{C}\mathrm{P}^n) \arrow[drr] & H^2(\mathbb{C}\mathrm{P}^n) \arrow[drr] & H^3(\mathbb{C}\mathrm{P}^n) \arrow[drr] & H^4(\mathbb{C}\mathrm{P}^n) \arrow[dr] & \cdots \\
0 & H^0(\mathbb{C}\mathrm{P}^n) & H^1(\mathbb{C}\mathrm{P}^n) & H^2(\mathbb{C}\mathrm{P}^n) & H^3(\mathbb{C}\mathrm{P}^n) & H^4(\mathbb{C}\mathrm{P}^n) & \cdots \\
 & 0 & 1 & 2 & 3 & 4 & p
\end{tikzcd}$$

Next observe that any differential into $E_2^{1,0} = H^1(\mathbb{C}P^n)$ has domain zero and any differential out of it has codomain zero, so $H^1(\mathbb{C}P^n) = E_\infty^{1,0} = \partial_{n=0}\mathbb{Z}$. But $E_2^{1,1} = E_2^{1,0}$, so $E_2^{1,1} = 0$ as well. But now that we have killed $E_0^{1,1}$, the differential $H^1(\mathbb{C}\mathrm{P}^n) \to H^3(\mathbb{C}\mathrm{P}^n)$ on the E_2-page must also be zero. Thus the same reason that we just used to show the $E_2^{1,\bullet}$-column is zero applies to the $E_2^{3,\bullet}$-column, and then the $E_2^{5,\bullet}$, and so on, meaning $E_2^{k,\bullet} = 0$ for all odd integers k.

The updated E_2-page is then the following.

$$\begin{tikzcd}
q & \vdots & \vdots & \vdots & \vdots & \vdots & \\
2 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & \mathbb{Z} \arrow[drr] & 0 & H^2(\mathbb{C}\mathrm{P}^n) \arrow[drr] & 0 & H^4(\mathbb{C}\mathrm{P}^n) \arrow[dr] & \cdots \\
0 & \mathbb{Z} & 0 & H^2(\mathbb{C}\mathrm{P}^n) & 0 & H^4(\mathbb{C}\mathrm{P}^n) & \cdots \\
 & 0 & 1 & 2 & 3 & 4 & p
\end{tikzcd}$$

Since only the bottom two rows are nonzero, the spectral sequence stabilizes at the E_3-page, i.e., $E_3^{p,q} = E_\infty^{p,q}$ for all $p, q \geq 0$. The E_∞-page of the Serre spectral sequence is given by $E_\infty^{p,q} = \delta_{p+q=2n+1}\mathbb{Z} + \delta_{p=0}\delta_{q=0}\mathbb{Z}$. We know that up to group extension, this means

$$E_3^{p,q} = E_\infty^{p,q} = H^{p+q}(S^{2n+1}) = \delta_{p+q=0}\mathbb{Z} + \delta_{p,q=0}\mathbb{Z}.$$

In particular, $E_3^{0,1} = \partial_{1+0=2n+1}\mathbb{Z} + \cancelto{0}{\partial_{1=0}\delta_{q=0}\mathbb{Z}} = \partial_{n=0}\mathbb{Z}$, so assuming $n \geq 1$ we must have

$\ker(\mathbb{Z} \to H^2(\mathbb{C}\mathrm{P}^n)) = 0$ for the $E_2^{1,0} = \mathbb{Z}$ term to vanish on the E_3-page. This means $\mathbb{Z} \to H^2(\mathbb{C}\mathrm{P}^2)$ is injective. This map must also be surjective, since otherwise $E_3^{2,0}$ would be nonzero, despite the fact $E_3^{2,0} = E_\infty^{2,0}$ equals, up to a group extension, $H^{2+0}(S^{2n+1}) = 0$, which is impossible since a group extension of a nonzero group must be nonzero. Thus $E_2^{2,0} = 0$, so because $E_{2,1} = E_{2,0}$ we have $E_{2,2} = \mathbb{Z}$, too.

We apply the same reasoning as we have for $q = 2$ above for $q = 4$ column, and then for the $q = 6$ column, and so on, until $q = 2n$. When $q = 2n$, since $H^{2n}(\mathbb{C}\mathrm{P}^n) = \mathbb{Z}$ because as a smooth manifold $\mathbb{C}\mathrm{P}^n$ has dimension $2n$ as a CW complex, the term $E_2^{2n,0} = \mathbb{Z}$-term is stable, so the map $\mathbb{Z} \to H^{2k+1} = 0$. Thus the E_2-page is zero in columns $\geq 2n+1$. We conclude that the complete E_2-page is the following.

$$\begin{tikzcd}
q & \vdots & \vdots & \vdots & \vdots & \vdots & \\
2 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & \mathbb{Z} \arrow[rrd, "\mathrm{id}"] & 0 & \mathbb{Z} \arrow[rrd, "\mathrm{id}"] & 0 & \mathbb{Z} \arrow[rd] & \cdots \\
0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & \cdots \\
 & 0 & 1 & 2 & 3 & 4 & p
\end{tikzcd}$$

This proves that $H^q(\mathbb{C}\mathrm{P}^n)$ is $\mathbb{Z}$ if $0 \leq q \leq 2n$ and q is even, and 0 otherwise.

We now compute the ring structure on the cohomology ring $H^\bullet(\mathbb{C}\mathrm{P}^n)$. Recall that each E_r-page is a ring with differentials ∂_r satisfying the Leibniz rule $\partial_r(ab) = \partial_r(a)b + (-1)^{|a|}a\partial_r(b)$ for all $a \in E_r^{p,q}$ and $b \in E_r^{s,t}$, where $|a| \coloneqq \deg(a) \coloneqq p+q$. Choose generators $a \in E_2^{0,1} = \mathbb{Z}$ and $b \in E_2^{2,0} = \mathbb{Z}$ that $\partial_2(ba) = b^2 \in E_2^{4,0}$ is a generator, so jumping up a row via multiplication by a we obtain a generator $ab^2 \in E_2^{4,1}$. Since ultimately $b^{n+1} \in E_2^{2n+2,0} = 0$, we conclude $H^\bullet(\mathbb{C}\mathrm{P}^n) = \mathbb{Z}[b]/(b^{n+1})$ with $|b| = 2$.

A.1.11. **The Wang and Gysin spectral sequences.** See here, here, and here.

A.1.12. **More examples of spectral sequences.** See here.

Bibliography

[Ark11] Martin Arkowitz. *Introduction to Homotopy Theory.* Universitext. Springer New York, New York, NY, 2011.

[Hat02] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, Cambridge, 2002.

[Hat17] Allen Hatcher. Vector bundles and k-theory, 2017. Lecture notes, Version 2.2, November 2017; contains basics on vector bundles and topological K-theory, available at `https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf`.

[HP12] Maximilien Holmberg-Péroux. The serre spectral sequence, 2012.

[May11] John Peter May. A concise course in algebraic topology, 2011.

[McC01] John McCleary. *A User's Guide to Spectral Sequences*, volume 58 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, 2nd edition, 2001.

[MP12] John Peter May and Kate Ponto. *More Concise Algebraic Topology: Localization, Completion, and Model Categories.* University of Chicago Press, 2012.

[Str11] Jeffrey Strom. *Modern Classical Homotopy Theory*, volume 127 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2011.