Algebraic higher categories

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Conventions

We read composition in 3D graphical calculus for $\boxtimes$ as back-to-front, for $\otimes$ as left-to-right, and for $\circ$ as bottom-to-top.

Contents

1. Categories, functors, natural transformations, and modifications

1.1. **Categories.** The following subsections in their totality should emphasize that one obtains a fully weak algebraic n-category as a "weakly enriched" fully weak algebraic $(n-1)$-category. A *monoidal n-category* is an $(n+1)$-category with exactly one object, often thought of as just an n-category with extra structure.

1.1.1. *0-categories.* A *0-category* A is a set.

1.1.2. *1-categories.* A *1-category* $\mathcal{C}$ consists of the following data.

- A collection of *0-cells* $a \in \mathcal{C}$.
- For each pair of 0-cells $a, b \in \mathcal{C}$, a 0-category $\mathcal{C}(a \to b)$ of *1-cells*.
- For each triple of 0-cells $a, b, c \in \mathcal{C}$, a *1-composition* 0-functor $\circ\colon \mathcal{C}(b \to c) \times \mathcal{C}(a \to b) \to \mathcal{C}(a \to c)$.
- For each 0-cell $a \in \mathcal{C}$, an *identity* 1-cell $\mathrm{id}_a \in \mathcal{C}(a \to a)$ (often also denoted a when clear).

These data are subject to the following conditions.

- The $\circ$ are strictly associative.
- The $\circ$ strictly preserve identity 1-cells in the left and right slots.

1.1.3. *2-categories.* A *2-category* $(\mathfrak{C}, \otimes, 1; \alpha, \lambda, \rho)$ consists of the following data.

- A collection of *0-cells* $a \in \mathfrak{C}$.
- For each pair of 0-cells $a, b \in \mathfrak{C}$, a *hom* 1-category $\mathfrak{C}(a \to b)$ of *1-cells* and *2-cells*.
- For each triple of 0-cells $a, b, c \in \mathfrak{C}$, a *1-composition* 1-functor $\otimes\colon \mathfrak{C}(a \to b) \times \mathfrak{C}(b \to c) \to \mathfrak{C}(a \to c)$.
- For each 0-cell $a \in \mathfrak{C}$, an *identity* 1-cell $1_a \in \mathfrak{C}(a \to a)$ (often also denoted a when clear).

These data are subject to the following conditions.

- The $\otimes$ are associative up to a 1-natural isomorphism $\alpha\colon -\otimes(-\otimes-) \overset{\cong}{\Rightarrow} (-\otimes-)\otimes-$ (the *associator*).
- The $\otimes$ preserve identity 1-cells in the left and right slots up to 1-natural isomorphisms $\lambda\colon 1_{(-)} \otimes - \overset{\cong}{\Rightarrow} -$ and $\rho\colon -\otimes 1_{(-)} \overset{\cong}{\Rightarrow} -$ (the *left* and *right unitors*) respectively.

The above are subject to the following coherence conditions.

- The component 1-cells of the following diagram strictly commute.

$$\begin{array}{ccccc}
-\otimes(-\otimes(-\otimes-)) & \overset{\alpha}{\Longrightarrow} & (-\otimes-)\otimes(-\otimes-) & \overset{\alpha}{\Longrightarrow} & ((-\otimes-)\otimes-)\otimes- \\
{\scriptstyle\alpha}\Downarrow & & & & \Uparrow{\scriptstyle\alpha} \\
-\otimes((-\otimes-)\otimes-) & & \overset{\alpha}{\Longrightarrow} & & (-\otimes(-\otimes-))\otimes-
\end{array}$$

- The component 1-cells of the following diagrams strictly commute.

$$\begin{array}{ccc}
1_{(-)}\otimes(-\otimes-) & & \\
{\scriptstyle\lambda}\Downarrow & \overset{\alpha}{\searrow} & (1_{(-)}\otimes-)\otimes- \\
-\otimes- & \overset{\lambda}{\swarrow} &
\end{array}
\qquad
\begin{array}{ccc}
(-\otimes-)\otimes 1_{(-)} & & \\
{\scriptstyle\rho}\Downarrow & \overset{\alpha}{\searrow} & -\otimes(-\otimes 1_{(-)}) \\
-\otimes- & \overset{\rho}{\swarrow} &
\end{array}$$

1.1.4. *3-categories.* A *3-category* $(\mathscr{C}, \boxtimes, \mathbf{1}; \alpha, \lambda, \rho; \pi, m, \ell, r)$ consists of the following data.

- A collection of *0-cells* $a \in \mathscr{C}$.
- For each pair of 0-cells $a, b \in \mathscr{C}$, a *hom* 2-category $\mathscr{C}(a \to b)$ of *1-cells*, *2-cells*, and *3-cells*.
- For each triplet of 0-cells $a, b, c \in \mathscr{C}$, a *1-composition* 2-functor $\boxtimes\colon \mathscr{C}(a \to b) \times \mathscr{C}(b \to c) \to \mathscr{C}(a \to c)$.
- For each 0-cell $a \in \mathscr{C}$, an *identity* 1-cell $\mathbf{1}_a \in \mathscr{C}(a \to a)$ (often also denoted a when clear).

These data are subject to the following conditions.

- The $\boxtimes$ are associative up to a 2-isomorphism $\alpha\colon -\boxtimes(-\boxtimes-) \overset{\cong}{\Rightarrow} (-\boxtimes-)\boxtimes-$ (the *associator*).
- The $\boxtimes$ preserve identity 1-cells in the left and right slots up to 2-isomorphisms $\lambda\colon \mathbf{1}_{(-)}\boxtimes - \overset{\cong}{\Rightarrow} -$ and $\rho\colon -\boxtimes\mathbf{1}_{(-)} \overset{\cong}{\Rightarrow} -$ (the *left* and *right 1-unitors*) respectively.

The above are subject to the following coherence conditions.

- The component 1-cells of the following pentagon commute up to a component 2-cell of an invertible 2-modification π (the *pentagonator*), as illustrated.

$$\begin{array}{ccccc}
-\boxtimes(-\boxtimes(-\boxtimes-)) & \overset{\alpha}{\Rightarrow} & (-\boxtimes-)\boxtimes(-\boxtimes-) & \overset{\alpha}{\Rightarrow} & ((-\boxtimes-)\boxtimes-)\boxtimes- \\
{\scriptstyle\alpha}\Downarrow & & \Downarrow{\scriptstyle\pi} & & \Uparrow{\scriptstyle\alpha} \\
-\boxtimes((-\boxtimes-)\boxtimes-) & & \overset{\alpha}{\Longrightarrow} & & (-\boxtimes(-\boxtimes-))\boxtimes-
\end{array}$$

- The component 1-cells of the following triangles commute up to a component 2-cell of invertible 2-modifications m, ℓ, and r (the *middle*, *left*, and *right 2-unitors*) respectively, as illustrated.

$$\begin{array}{ccc} 1_{(-)} \boxtimes (- \boxtimes -) & & \\ \lambda \Downarrow & \overset{\ell}{\Rrightarrow} \;\; \overset{\alpha}{\searrow} & (1_{(-)} \boxtimes -) \boxtimes - \\ - \boxtimes - & \underset{\lambda}{\swarrow} & \end{array} \qquad \begin{array}{ccc} (- \boxtimes 1_{(-)}) \boxtimes - & & \\ \rho \Downarrow & \overset{m}{\Rrightarrow} \;\; \overset{\alpha}{\searrow} & (- \boxtimes 1_{(-)}) \boxtimes - \\ - \boxtimes - & \underset{\lambda}{\swarrow} & \end{array} \qquad \begin{array}{ccc} (- \boxtimes -) \boxtimes 1_{(-)} & & \\ \rho \Downarrow & \overset{r}{\Rrightarrow} \;\; \overset{\alpha}{\searrow} & - \boxtimes (- \boxtimes 1_{(-)}) \\ - \boxtimes - & \underset{\rho}{\swarrow} & \end{array}$$

- The *non-abelian 4-cocycle condition* [JY21, 11.2.14, p. 326][1] is satisfied.
- The *left* and *right normalization conditions* [JY21, 11.2.16–7, p. 327][2] are satisfied.

1.2. **Functors.** An n-functor is a morphism of n-categories.

1.2.1. *0-functors.* A 0-*functor* is a set function $f \colon A \to B$.

1.2.2. *1-functors.* A *1-functor* $F \colon \mathcal{C} \to \mathcal{D}$ consists of the following data.

- For each 0-cell $a \in \mathcal{C}$, of a 0-cell $F(a) \in \mathcal{D}$.
- For each pair of 0-cells $a, b \in \mathcal{C}$, a *local hom* 0-functor $F \colon \mathcal{C}(a \to b) \to \mathcal{D}(F(a) \to F(b))$.

These data are subject to the following conditions.

- The local hom 0-functors strictly preserve $\circ$.
- The local hom 0-functors F strictly preserve identity 1-cells.

1.2.3. *2-functors.* A *2-functor* $F \colon \mathfrak{C} \to \mathfrak{D}$ consists of the following data.

- For each 0-cell $a \in \mathfrak{C}$, a 0-cell $F(a) \in \mathfrak{D}$.
- For each pair of 0-cells $a, b \in \mathcal{C}$, a *local hom* 1-functor $F \colon \mathfrak{C}(a \to b) \to \mathfrak{D}(F(a) \to F(b))$.

These data are subject to the following conditions.

- The local hom 1-functors preserve $\otimes$ up to a 1-natural isomorphism $\mu \colon F(-) \otimes F(-) \overset{\cong}{\Rightarrow} F(- \otimes -)$ (the *1-compositor*).
- The local hom 1-functors preserve identity 1-cells up to a 1-natural isomorphism $\nu \colon F(\mathrm{id}_{(-)}) \overset{\cong}{\Rightarrow} \mathrm{id}_{F(-)}$ (the *1-unitor*).[3]

The above are subject to the following coherence conditions.

- The following diagram strictly commutes in $\mathfrak{D}$.

$$\begin{array}{ccccc} F(-) \otimes (F(-) \otimes F(-)) & \overset{\mu}{\Longrightarrow} & F(-) \otimes F(- \otimes -) & \overset{\mu}{\Longrightarrow} & F(- \otimes (- \otimes -)) \\ \Downarrow \alpha & & & & \Downarrow F(\alpha) \\ (F(-) \otimes F(-)) \otimes F(-) & \overset{\mu}{\Longrightarrow} & F(- \otimes -) \otimes F(-) & \overset{\mu}{\Longrightarrow} & F((- \otimes -) \otimes -) \end{array}$$

- The following diagrams strictly commute in $\mathfrak{D}$.

$$\begin{array}{ccc} 1_{F(a)} \otimes F(-) & \overset{\lambda}{\Longrightarrow} & F(-) \\ \nu \Downarrow & & \Uparrow F(\lambda) \\ F(1_a) \otimes F(-) & \overset{\mu}{\Longrightarrow} & F(1_a \otimes -) \end{array} \qquad \begin{array}{ccc} F(-) \otimes 1_{F(a)} & \overset{\rho}{\Longrightarrow} & F(-) \\ \nu \Downarrow & & \Uparrow F(\rho) \\ F(-) \otimes F(1_a) & \overset{\mu}{\Longrightarrow} & F(- \otimes 1_a) \end{array}$$

1.3. **Natural transformations.** An *n-natural transformation* is a morphism of n-functors.

1.3.1. *1-natural transformations.* A *1-natural transformation* $\eta\colon F \Rightarrow G$ consists of the following data.

- For each 0-cell $a \in \mathcal{C}$, a 1-cell $\eta_a \in \mathcal{D}(F(a) \to G(a))$.

These data are subject to the following conditions.

- η is strictly compatible with 1-composition $\otimes$, i.e., for all $f \in \mathcal{C}(a \to b)$, the following diagram strictly commutes in $\mathcal{D}$.

$$\begin{CD} F(a) @>{F(f)}>> F(b) \\ @V{\eta_a}VV @VV{\eta_b}V \\ G(a) @>{G(f)}>> G(b) \end{CD}$$

1.3.2. *2-natural transformations.* A *2-natural transformation* $\eta\colon F \Rightarrow G$ consists of the following data.

- For each 0-cell $a \in \mathfrak{C}$, a 1-cell $\eta_a \in \mathfrak{D}(F(a) \to G(a))$.

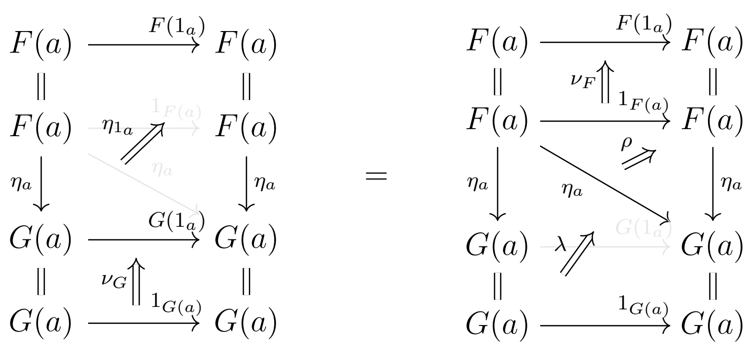
These data satisfy the following conditions.

- η is compatible with 1-composition $\otimes$ up to an invertible 2-cell η_f of the following type.[4]

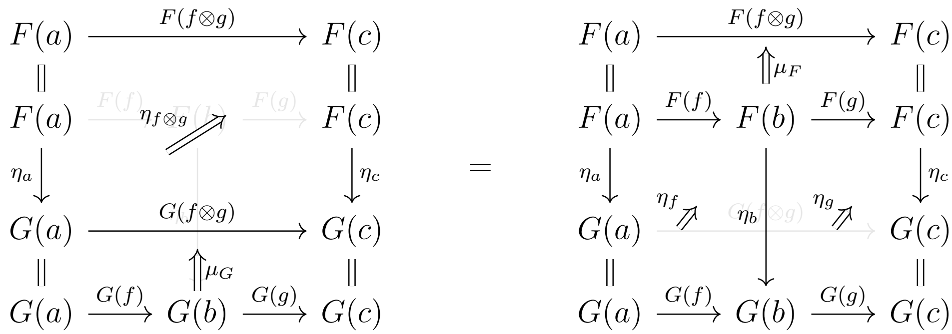
$$\begin{CD} F(a) @>{F(f)}>> F(b) \\ @V{\eta_a}VV \overset{\eta_f}{\nearrow\!\!\!\nearrow} @VV{\eta_b}V \\ G(a) @>{G(f)}>> G(b) \end{CD}$$

The above are subject to the following coherence conditions.

- For all 0-cells $a \in \mathfrak{C}$, the following equation holds.



- For all 0-cells $a, b, c \in \mathfrak{C}$ and 1-cells $f \in \mathfrak{C}(a \to b)$ and $g \in \mathfrak{C}(b \to c)$, the following equation holds.

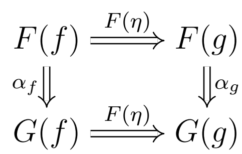


1.4. **Icons.** We call parallel 2-functors $F, G\colon \mathfrak{C} \to \mathfrak{D}$ *iconic* if $F = G$ on 0-cells and there is an *icon* $\alpha\colon F \Rightarrow G$, that is, an identity-component oplax 2- natural transformation. Thus an *icon* consists of the following data.

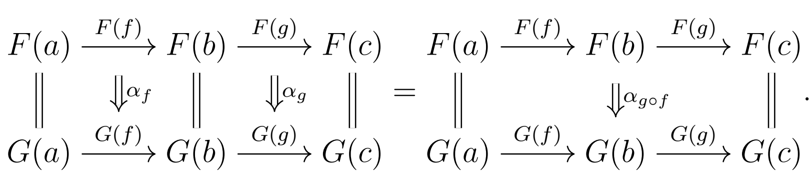
- For each 1-cell $f \in \mathfrak{C}(a \to b)$, a 2-cell[5] $\alpha_f \in \mathfrak{D}(F(f) \Rightarrow G(f))$.

These data satisfy the following conditions.

- For each 2-morphism $\eta \in \mathfrak{C}(f \Rightarrow g)$, the following diagram strictly commutes in $\mathfrak{D}$.



- For each object $a \in \mathfrak{C}$, $\alpha_{1_a} \in \mathfrak{D}(F(f) \Rightarrow G(f))$ is the identity 2-cell id_{1_a} (up to the unitors of F and G, which I will omit here for clarity).



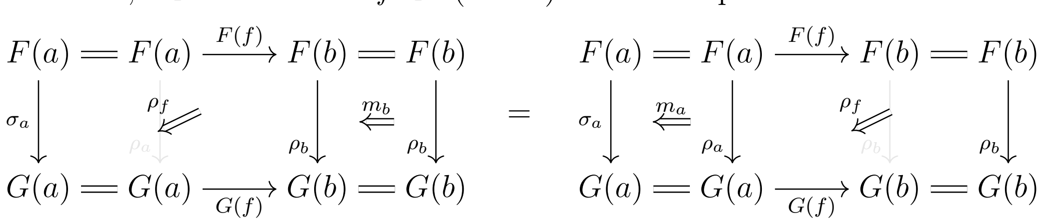
1.5. **Modifications.** An n-modification is a morphism of n-natural transformations.

1.5.1. *2-modifications.* A *2-modification* $m\colon \rho \Rrightarrow \sigma$ consists of the following data.

- For each 0-cell $a \in \mathfrak{C}$, a 2-cell $m_c\colon \rho_c \Rightarrow \sigma_c$.

These data are subject to the following conditions.

- For all 0-cells $a, b \in \mathfrak{C}$ and 1-cells $f \in \mathfrak{C}(a \to b)$ commute up to 2-natural transformation



1.6. **Equivalences.**

1-equivalences. A *1-equivalence* of 1-categories is a 1-functor F that is invertible up to 2-natural isomorphism. Equivalently, this means F is essentially surjective on 0-cells and fully faithful on 1-cells.[6]

2-equivalences. A *2-equivalence* of 2-categories is a 2-functor F that is invertible up to 2-natural equivalence.[7] Equivalently, this means F is essentially surjective on 0-cells and fully faithful on 0- and 1-cells.[8]

1.7. **Skeletal categories.**

Skeletal 1-categories. A 1-category is *skeletal* if it has no non-identity isomorphisms. The *skeleton* of a 1-category is a skeletal subcategory whose inclusion is an equivalence.[9]

Skeletal 2-categories.[10] A 2-category $\mathfrak{C}$ is *1-skeletal* if the hom categories $\mathfrak{C}(x \to y)$ are skeletal for each pair of objects $x, y \in \mathfrak{C}$; $\mathfrak{C}$ is *0-skeletal* if x is equivalent to y if and only if $x = y$. Finally, $\mathfrak{C}$ is *skeletal* if it is both 0-skeletal and 1-skeletal. A symmetric monoidal 2-category is k-skeletal if it underlying 2-category is k-skeletal.[11]

In what follows, the n-th strictification mentioned is the "best" (currently known) faithful strictification of the corresponding algebraic fully weak n-categories above. Note 0- and 1-categories are strict as defined, and that every 2-category is equivalent to a strict 2-category. Often, *semistrict n-category* refers to the "most strict" notion of n-category that is still equivalent to the notion of fully weak n-category.

1.8. **Semistrict 3-categories.** Unfortunately, for a general 3-category $\mathscr{C}$, it is possible that $\mathscr{C}$ is not equivalent to any strict 3-category. However, [Gur13, Corollary 9.15] proves $\mathscr{C}$ is equivalent to a *semistrict* 3-category (i.e., a strict and cubical 3-category, i.e., the Gray-enriched (1-)category), where Gray is the monoidal 1-category of strict 2- categories and strict 2-functors with the Gray monoidal structure, which is the universal monoidal structure $\boxtimes$ for which $2\mathsf{Cat}_{\text{str}}(\mathscr{C} \boxtimes \mathscr{D} \to \mathscr{E}) \simeq 2\mathsf{Cat}_{\text{str}}(\mathscr{C} \to 2\mathsf{Fun}_{\text{str}}(\mathscr{D} \to \mathscr{E})))$. That is, any 3-category is equivalent to a *semistrict 3-category* $(\mathscr{C}, (-)_{\boxtimes}, (-)^{\boxtimes}, \mathbf{1}, \phi)$, which consists of the following data.[12]

- A collection of *0-cells* $a \in \mathscr{C}$.
- For each pair of 0-cells $a, b \in \mathscr{C}$, a strict *hom* 2-category $\mathscr{C}(a \to b)$ of *1-cells*, *2-cells*, and *3-cells*.
- For each 0-cell $c \in \mathscr{C}$ and each 1-cell $f \in \mathscr{C}(a \to b)$, a *1-postcomposition* strict 2-functor $f \boxtimes (-) =: f_{\boxtimes} \colon \mathscr{C}(b \to c) \to \mathscr{C}(a \to c)$.
- For each 0-cell $a \in \mathscr{C}$ and each 1-cell $g \in \mathscr{C}(b \to c)$, a *1-precomposition* strict 2-functor $(-) \boxtimes g =: g^{\boxtimes} \colon \mathscr{C}(a \to b) \to \mathscr{C}(a \to c)$.
- For each 0-cell $a \in \mathscr{C}$, an *identity* 1-cell $\mathbf{1}_a \in \mathscr{C}(a \to a)$.
- For each pair of 2-cells $\gamma \in \mathscr{C}({}_a f_b \Rightarrow {}_a f'_b)$, $\xi \in \mathscr{C}({}_b g_c \Rightarrow {}_b g'_c)$, an invertible 3-cell $\phi_{\gamma,\xi} \in \mathscr{C}((\xi \boxtimes f') \otimes (g \boxtimes \gamma) \Rrightarrow (g' \boxtimes \gamma) \otimes (\xi \boxtimes f))$ (the *interchanger*).

The above data are subject to the following conditions.

- For composable 1-cells f, g in $\mathscr{C}$, $f_{\boxtimes}(g) = g^{\boxtimes}(f) =: f \boxtimes g$.
- The $(-)_{\boxtimes}$ and $(-)^{\boxtimes}$ strictly preserve 1-composition, i.e., $(- \boxtimes -)_{\boxtimes} = (-)_{\boxtimes} \boxtimes (-)_{\boxtimes}$ and $(- \boxtimes -)^{\boxtimes} = (-)^{\boxtimes} \boxtimes (-)^{\boxtimes}$.
- The $(-)_{\boxtimes}$ and $(-)^{\boxtimes}$ strictly preserve identity 1-cells, i.e., $- \boxtimes \mathbf{1}_{(-)} = \mathbf{1}_{\mathscr{C}(a\to b)} = \mathbf{1}_{(-)} \boxtimes -$.
- The $\phi_{-,-}$ strictly preserve identity 2-cells in each slot, i.e., for each 1-cell $f \in \mathscr{C}(a \to b)$, $\phi_{-,1_f} = - \boxtimes 1_f$ and $\phi_{1_f,-} = 1_f \boxtimes -$.
- The ϕ strictly preserve the $\otimes$, i.e., for $g \overset{\xi}{\Rightarrow} g' \overset{\xi'}{\Rightarrow} g''$ and $f \overset{\gamma}{\Rightarrow} f' \overset{\gamma'}{\Rightarrow} f''$, the following hold whenever they make sense.
$$\phi_{\xi'\otimes\xi,\gamma} = (\phi_{\xi',\gamma} \otimes (\xi \otimes f)) \circ ((\xi' \otimes f') \otimes \phi_{\xi,\gamma}),$$
$$\phi_{\xi,\gamma\otimes\gamma'} = ((g' \otimes \gamma') \otimes \phi_{\xi,\gamma}) \circ (\phi_{\xi,\gamma'} \otimes (g \otimes \gamma)).$$
- The $\phi_{-,-}$ are natural, i.e., for 1-cells $g, g' \in \mathscr{C}(b \to c)$, 2-cells $\xi, \xi' \in \mathscr{C}(g \Rightarrow g')$, and a 3-cell $\Xi \in \mathscr{C}(\xi \Rrightarrow \xi')$; and for 1-cells $f, f' \in \mathscr{C}(a \to b)$, $\gamma, \gamma' \in \mathscr{C}(f \Rightarrow f')$, and a 3-cell $\Gamma \in \mathscr{C}(\gamma \Rrightarrow \gamma')$, we have $\phi_{\xi',\gamma} \circ ((\Xi \otimes f') \otimes \mathrm{id}_{g\otimes\gamma}) = (\mathrm{id}_{g'\otimes\gamma} \otimes (\Xi \otimes f)) \circ \phi_{\xi,\gamma}$ and $\phi_{\xi,\gamma'} \circ (\mathrm{id}_{\xi\otimes f'} \otimes (g \otimes \Gamma)) = ((g' \otimes \Gamma) \otimes \mathrm{id}_{\xi\otimes f}) \circ \phi_{\xi,\gamma}$.
- The $\phi_{-,-}$ strictly respect the $(-)_{\boxtimes}$ and $(-)^{\boxtimes}$, i.e., for 1-morphisms f, g, h and 2-morphisms σ, ξ, γ, the equations $\phi_{h\boxtimes\xi,\gamma} = h \boxtimes \phi_{\xi,\gamma}$, $\phi_{\sigma\boxtimes g,\gamma} = \phi_{\sigma,\gamma} \boxtimes g = \phi_{\sigma,g\boxtimes\gamma}$, and $\phi_{\sigma,\xi\boxtimes f} = \phi_{\sigma,\xi} \boxtimes f$ hold when they make sense.

2. Enrichment

Fix a monoidal category $(\mathcal{V}, \otimes_{\mathcal{V}}, 1_{\mathcal{V}})$. (It is helpful to think of $(V, \otimes_{\mathcal{V}}, 1_{\mathcal{V}})$ as $(\mathsf{Vec}, \otimes_{\mathbb{C}}, \mathbb{C})$.)

2.1. **Enriched 0-categories.** A *$\mathcal{V}$-enriched 0-category* is a set A of elements of $\mathcal{V}$. Thus a 0-category is a $(\mathsf{Set}, \times, \mathrm{pt})$-enriched 0-category.

2.2. **Enriched 1-categories.** A *$\mathcal{V}$-enriched 1-category* consists of the following data.

- A collection of *0-cells* $a \in \mathcal{C}$.
- For each pair of 0-cells $a, b \in \mathcal{C}$, a *hom* 0-cell $\mathcal{C}(a \to b) \in \mathcal{V}$; its *elements* or *1-cells* $f \in \mathcal{C}(a \to b)$ are 1-cells $f \in \mathcal{V}(1_{\mathcal{V}} \to \mathcal{C}(a \to b))$.
- For each triplet of 0-cells $a, b, c \in \mathcal{C}$, a *1-composition* 1-cell $\circ \in \mathcal{V}(\mathcal{C}(b \to c) \otimes_{\mathcal{V}} \mathcal{C}(a \to b) \to \mathcal{C}(a \to b))$.
- For each 0-cell $a \in \mathcal{C}$, an *identity* element $\jmath_a \in \mathcal{V}(1_{\mathcal{V}} \to \mathcal{C}(a \to a))$.

These data are subject to the following conditions.

- The $\circ$ are strictly associative.
- The $\circ$ strictly preserve identity 1-cells in the left and right slots.

2.2.1. *From a module category to an enriched category.* From a finite semisimple left $\mathcal{V}$-module category $\mathcal{M}$, one can build a $\mathcal{V}$-enriched category $\widehat{\mathcal{M}}$ by showing $\mathcal{M}(- \triangleright m \to n)\colon \mathcal{V}^{\mathrm{op}} \to \mathsf{Set}$ is representable and defining the *internal hom* object $\widehat{\mathcal{M}}(m \to n) \in \mathcal{V}$ to be its representing object.

2.3. **Enriched 0-functors.**
2.4. **Enriched 1-functors.** Fix a monoidal 1-category $(\mathcal{V}, \otimes_{\mathcal{V}}, 1_{\mathcal{V}})$. A *$\mathcal{V}$-enriched 1-functor* $F\colon \mathcal{C} \to \mathcal{D}$ consists of the following data.

- For each 0-cell $a \in \mathcal{C}$, of a 0-cell $F(a) \in \mathcal{D}$.
- For each pair of 0-cells $a, b \in \mathcal{C}$, a *local hom* $\mathcal{V}$-enriched 0-functor $F \in \mathcal{V}(\mathcal{C}(a \to b) \to \mathcal{D}(F(a) \to F(b))$.

These data are subject to the following conditions.

- The local hom 0-functors strictly preserve $\otimes$.
- The local hom 0-functors strictly preserve identity 1-cells.

Thus a $\mathcal{V}$-enriched 1-functor of $\mathcal{V}$-enriched 1-categories is simply a functor in the usual sense of the underlying categories.[13]
2.5. **TODO: Tensored vs. Enriched vs. Module categories.** **TODO: I should follow Kelly05 for this**

3. Toward multitensor categories

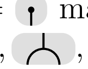
3.1. **Linear categories.**
3.1.1. *Linear 1-categories.* A 1-category $\mathcal{C}$ is called *linear* if it is locally linear[14] and 1-composition $\circ$ is bilinear.[15]
3.1.2. *Linear 2-categories.* A 2-category $\mathfrak{C}$ is called *linear* if it is locally linear and 1-composition $\otimes$ is bilinear.
3.2. **Linking algebras.**
3.2.1. *Linking 1-algebras.* Given n objects $x_1, \ldots, x_n \in \mathcal{C}$, the *$n$-fold linking algebra* $L(x_1, \ldots, x_n)$ is the algebra $\bigoplus_{i,j=1}^{r} \mathcal{C}(x_j \to x_i) = \left[\mathcal{C}(x_j \to x_i)\right]_{(i,j)}$ whose elements are formal matrices with product given by matrix multiplication.[16][17]
3.2.2. *Linking 2-algebras.* **TODO:** A *linking E_1-algebra* is an algebra in Vec over the ($\mathsf{Top} = \mathcal{T} =$ CGWH-enriched) E_1-operad.
3.3. **Presemisimple categories.**
3.3.1. *Presemisimple 1-categories.* A linear 1-category $\mathcal{C}$ is called *presemisimple* if all n-fold linking algebras are finite-dimensional and semisimple. (If $\mathcal{C}$ admits finite direct sums, then this is equivalent to the endomorphism algebras being finite-dimensional and semisimple.)
A presemisimple 1-category is *finite* if $\mathrm{Irr}(\mathcal{C})$ is a finite set.
3.3.2. *Presemisimple 2-categories.* A linear 2-category $\mathfrak{C}$ is *presemisimple* if all n-fold linking algebras are semisimple multitensor categories.
A presemisimple 2-category is *finite* if all n-fold linking algebras are multi*fusion* and there is a global bound on the dimensions of $\mathrm{End}(1)$ for the centers of all linking algebras.[18]
3.4. **Additive completeness.**

3.4.1. *1-additive completeness.* A 1-category $\mathcal{C}$ is additive complete if $\mathcal{C}$ admits all finite direct sums (i.e., $\mathcal{C}$ is *additive complete*).

3.4.2. *2-additive completeness.*

3.5. Idempotent completeness.

3.5.1. *1-idempotent completeness.* In a 1-category $\mathcal{C}$, a *1-idempotent* (c, e) consists of an object $c \in \mathcal{C}$ and a morphism $e\colon c \to c$ with $e \circ e = e$. A 1-idempotent (c, e) *splits* if there is an object $a \in \mathcal{C}$ and morphisms $a \xrightarrow{i} c \xrightarrow{p} a$ such that $p \circ i = \mathrm{id}_a$ and $i \circ p = e$. We call $\mathcal{C}$ *idempotent complete* if all 1-idempotents split.[19]

3.5.2. *2-idempotent completeness.* For a 2-category $\mathfrak{C}$, a (unital) *algebra*[20] (A, μ, ι) in $\mathrm{End}(a)$ consists of a 1-morphism $A =$ $\in \mathfrak{C}(a \to a)$ and 2-morphisms $\mu =$  $\in \mathfrak{C}(A \otimes A \Rightarrow A)$ ("multiplication") and $\iota =$ $\in \mathfrak{C}(1_A \Rightarrow A)$ ("the unit") satisfying associativity ( = ) and unitality ( = = ). We call an algebra (A, μ, ι) *separable* when equipped with an (A, A)-bimodule map $\Delta =$  $\in \mathfrak{C}(A \Rightarrow A \otimes A)$ (meaning  =  = ) that splits μ (meaning  =). An algebra $(A, \mu, \iota, \Delta) =$ (, , , ) is called *Frobenius* if Δ admits a counit $\varepsilon =$ making (A, Δ, ε) a (counital) coalgebra[21]. A *condensation algebra* $(A, \mu, \iota, \Delta, \varepsilon) =$ (, , , , ) is just a separable Frobenius algebra. A *separable adjunction* for a 1-morphism $X \in \mathfrak{C}(a \to b)$ consists of a 1-morphism $X^\vee \in \mathfrak{C}(b \to a)$ and 2-morphisms $\mathrm{ev} \in \mathfrak{C}(X^\vee \otimes X \Rightarrow 1_b)$ and $\mathrm{coev} \in \mathfrak{C}(1_a \Rightarrow X \otimes X^\vee)$ satisfying the zig-zag equations and such that ev admits a right inverse $\varepsilon \in \mathfrak{C}(1_b \Rightarrow X^\vee \otimes X)$. For any separable adjunction $X \dashv X^\vee$, we can canonically endow $X \otimes X^\vee$ with the structure of a unital condensation algebra. Indeed, one can just check that $\mu =$ , $\Delta =$ , $\iota =$ , and $\varepsilon =$  works. A condensation algebra $(A, \mu, \iota, \Delta, \varepsilon) =$ (, , , , ) *splits* if it is isomorphic to such a condensation algebra $X \otimes X^\vee$ as algebras in $\mathrm{End}_{\mathfrak{C}}(a)$.[22]

A 2-category $\mathfrak{C}$ is said to be *2-idempotent complete* (or *condensation complete*) if all condensation algebras split.

3.6. Cauchy completeness.

3.6.1. *1-Cauchy completeness.* A 1-category is *1-Cauchy complete* if it has all direct sums and all idempotents split.

3.6.2. *2-Cauchy completeness.* A 2-category is *2-Cauchy complete* if it has finite 2-direct sums and all 2-idempotents split.

3.7. Semisimplicity.

3.7.1. *1-semisimplicity.* A presemisimple 1-category $\mathcal{C}$ is *semisimple* if it is 1-Cauchy complete.[23]

3.7.2. *2-semisimplicity.* A presemisimple 2-category $\mathcal{C}$ is *semisimple* if it is 2-Cauchy complete.

3.8. Rigidity.

3.8.1. *1-rigidity.* In a monoidal 1-category $(\mathcal{C}, \otimes, 1)$, an object c^* is a *right dual*,[24] of an object c, or equivalently c is a *left dual* of c^*, if there are 1-morphisms $\mathrm{ev} \in \mathcal{C}(c^* \otimes c \to 1) \otimes c$

and coev $\in \mathcal{C}(1 \to c \otimes c^*)$ satisfying the snake equations $(c \otimes \mathrm{ev}) \circ (\mathrm{coev} \otimes c) = \mathrm{id}_c$ and $(\mathrm{ev} \otimes c^*) \circ (c^* \otimes \mathrm{coev}) = \mathrm{id}_{c^*}$. We call $\mathcal{C}$ *rigid* if each 0-cell admits a left and right dual.

3.8.2. *2-rigidity.* In a monoidal 2-category $(\mathfrak{C}, \boxtimes, \mathbf{1})$, an object $c^\#$ is a *right dual* of an object c, or equivalently c is a *left dual* of $c^\#$, if there are 1-morphisms[25] $\mathrm{ev} \in \mathfrak{C}(c^\# \boxtimes c \to \mathbf{1})$ and $\mathrm{coev} \in \mathfrak{C}(\mathbf{1} \to c \boxtimes c^\#)$ satisfying the snake equations up to coherent[26] invertible 2-cells[27] $\mathrm{cusp} \in \mathfrak{C}((\mathrm{coev} \boxtimes c) \otimes (c \boxtimes \mathrm{ev}) \Rightarrow 1_c)$ and $\mathrm{cocusp} \in \mathfrak{C}(1_{c^\#} \Rightarrow (c^\# \boxtimes \mathrm{coev}) \otimes (\mathrm{ev} \boxtimes c^\#))$. We call $\mathfrak{C}$ *rigid* if each 0-cell admits a left dual and a right dual and each 1-cell admits a left adjoint and a right adjoint.[28][29]

3.9. **(Multi)tensor and (multi)fusion categories.**

3.9.1. *(Multi)tensor and (multi)fusion 1-categories.* A *multitensor category* is a rigid semisimple linear monoidal category $(\mathcal{C}, \otimes, 1)$.[30] A *tensor category* (AKA *infusion category*) is a multitensor category with simple monoidal unit 1. A *(multi)fusion category* is a finite (multi)tensor category.

3.9.2. *(Multi)tensor and (multi)fusion 2-categories.* A *(pre)fusion 2-category* is a finite (pre)semisimple rigid linear monoidal 2-category with simple monoidal unit $\mathbf{1}$.[31]

3.10. **Dual functors.**

3.10.1. *Dual 1-functors.* When a multitensor 1-category $\mathcal{C}$ is rigid, a choice of duality data $(c^\vee, \mathrm{ev}_c, \mathrm{coev}_c)$ for every $c \in \mathcal{C}$ assembles into a *dual 1-functor*, which is a monoidal 1-functor $\vee \colon \mathcal{C} \to \mathcal{C}^{\otimes\mathrm{op},\circ\mathrm{op}}$ given by "180 degree rotation" of 1-cells using "cups" (coev's) and "caps" (ev's). For any monoidal functor between rigid multitensor categories with a choice of dual functor, $F \colon (\mathcal{C}, \vee^{\mathcal{C}}) \to (\mathcal{D}, \vee^{\mathcal{D}})$, we have a canonical natural isomorphism $F \circ \vee \Rightarrow \vee \circ F$ given here, Def. 2.13, p. 30.

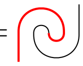
3.10.2. *Dual 2-functors.* The obvious first choice of generalization of dual 1-functor from multitensor 1-categories $(\mathcal{C}, \otimes, 1)$ to multitensor 2-categories $(\mathcal{C}, \boxtimes, \mathbf{1})$ is to ask that all the hom 1-categories are equipped with a dual functor (now called an adjoint functor by a previous footnote) that is coherently compatible with $\boxtimes$. However, we could instead ask that the objects of $\mathfrak{C}$ have duals whose evaluation and coevaluation 1-morphisms satisfy snake equations up to an appropriate coherence condition. To avoid confusion, we will refer to the former of these (that is, to a choice of adjoints for 1-morphisms) as *adjoint 2-functor* or alternatively as a *planar dual 2-functor*, and use *transverse dual 2-functor* for the latter. We will then define a *dual 2-functor*[32] to be a choice of planar dual 2-functor together with a choice of transverse dual 2-functor that satisfy compatibility conditions between each other. Note that our terminology here is nonstandard.

Object dual 2-functor. When a multitensor 2-category is rigid[33], a choice of dual $(c^\#, \mathrm{ev}, \mathrm{coev}, \mathrm{cusp}, \mathrm{cusp}^\#)$ for every $c \in \mathcal{C}$ assembles into a *dual 2-functor*, which is a monoidal 2-functor $\# \colon \mathfrak{C} \to \mathfrak{C}^{\boxtimes\mathrm{op},\otimes\mathrm{op}}$ that sends c to $c^\#$ for each $c \in \mathcal{C}$.

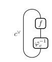
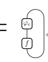
TODO: A *dual 2-functor*

Planar 2-functor. When a multitensor 2-category is rigid,[34] a choice of adjoint $(f^\vee, \mathrm{ev}, \mathrm{coev})$ for every 1-cell f that preserves $\boxtimes$ up to coherent isomorphism[35] in $\mathfrak{C}$ assembles into a *planar dual 2-functor*, which is a monoidal 2-functor $\#\colon \mathfrak{C} \to \mathfrak{C}^{\boxtimes\mathrm{op},\otimes\mathrm{op}}$ TODO: . A *dual 2-functor* TODO:

3.11. **Pivotal structures.**

3.11.1. *1-pivotal structures.* A *pivotal structure* $(\vee, \varphi)$ on a multitensor 1-category $\mathcal{C}$ is a choice of dual functor $\vee\colon \mathcal{C} \to \mathcal{C}^{\otimes\mathrm{op},\circ\mathrm{op}}$ together with a monoidal natural isomorphism $\varphi\colon \mathrm{id}_{\mathcal{C}} \Rightarrow \vee \circ \vee$. Two pivotal structures $(\vee_1, \varphi^1)$ and $(\vee_2, \varphi^2)$ are *equivalent* if $\varphi^1 =$  $\circ\, \varphi^2$. When a multitensor category $\mathcal{C}$ admits a pivotal structure, the set of pivotal structures is a torsor[36] for $\mathrm{Aut}_\otimes(\mathrm{id}_{\mathcal{C}}) \cong \mathrm{Hom}(\mathcal{U}_{\mathcal{C}} \to \mathbb{C}^\times)$.

A monoidal functor between pivotal categories $F\colon (\mathcal{C}, \vee^{\mathcal{C}}, \varphi^{\mathcal{C}}) \to (\mathcal{D}, \vee^{\mathcal{D}}, \varphi^{\mathcal{D}})$ is called *pivotal* if $\delta_c^\vee \circ \varphi^{\mathcal{D}}_{F(c)} = \delta_{c^\vee} \circ F(\varphi^{\mathcal{C}}_c)$ for all $c \in \mathcal{C}$.[37] Pivotal functors preserve left/right quantum traces. By **TODO: Pen20, Proposition 3.45**, F is pivotal if and only if δ_c is unitary for all $c \in \mathcal{C}$.

Traces and quantum dimension. A pivotal multitensor 1-category $(\mathcal{C}, \vee, \varphi)$ admits $\mathrm{End}(1_{\mathcal{C}})$-valued traces $\mathrm{tr}_{L/R}\colon \mathrm{End}(c) \to \mathrm{End}(1_{\mathcal{C}})$ given by $\mathrm{tr}^\varphi_L(f) :=$  and $\mathrm{tr}^\varphi_R(f) :=$  called the *left/right quantum trace*. If $c \in \mathcal{C}$ is simple, then where $1_{\mathcal{C}} = \oplus_{k=1}^r 1_k$, we have $c = 1_{\mathcal{C}} \otimes c \otimes 1_{\mathcal{C}} = \bigoplus_{i,j=1}^r (1_i \otimes s \otimes 1_j) = 1_{s(c)} \otimes c \otimes 1_{t(c)}$, we define the left/right *quantum dimension* as $\dim^{\vee,\varphi}_{L/R}(c) := \mathrm{tr}_{L/R}(p_{s(t)} \otimes \mathrm{id}_c \otimes p_{t(c)})$ where $p_k \in \mathrm{End}(1_{\mathcal{C}})$ is the projection obtained by splitting the idempotent id_{1_k}. Note that $\dim_L(c) = \dim_R(c^\vee)$. If $(\mathcal{C}, \vee, \varphi)$ is *pseudounitary*, meaning all quantum dimensions are positive, then $\dim^\varphi_L(c) = \dim^\varphi_R(c) = \mathrm{FPdim}(c)$ for all simple $c \in \mathcal{C}$.

3.11.2. *2-pivotal structures.*

Planar pivotal structures. A *planar pivotal structure* $(\varphi, \vee, \#)$ on a multitensor 2-category $(\mathfrak{C}, \otimes, \mathbf{1})$ consists of a monoidal 2-natural isomorphism $\varphi\colon \mathrm{id}_{\mathrm{id}_{\mathfrak{C}}} \Rightarrow \#$. A *planar pivotal* 2-category is a multitensor 2-category equipped with a monoidal TODO: See **here**.

A *pivotal structure* on a 2-category TODO: See **here**.

3.12. **Sphericality.**

3.12.1. *1-sphericality.* A *spherical multitensor 1-category* is a pivotal multitensor category $(\mathcal{C}, \vee, \varphi)$ such that for every simple object $c \in \mathcal{C}$, $\dim^{\vee,\varphi}_L(c) = \dim^{\vee,\varphi}_R(c)$. Thus for example pseudounitary pivotal multitensor categories are spherical.

3.12.2. *2-sphericality.* A *spherical 2-category* is a pivotal 2-category such that the front and back 2-spherical traces agree. TODO: See **here**.

4. Module structures

5. Dagger (and other) structures

5.1. **Dagger 0-categories.** A *dagger 0-category* is a $*$-*algebra* A, i.e., a finite-dimensional associative algebra A with a strict *anti-involution*, that is, with a map $(-)^*: A \to A^{\mathrm{op}}$ such that $(ab)^* = b^*a^*$ for all $a, b \in A$, $(a^*)^* = a$ for all $a \in A$, and $1^* = 1$.

A C*-*0-category* is a C*-*algebra*, i.e., a Banach (meaning Cauchy complete in the norm) $*$-algebra A satisfying the C*-*identity* $\|x^*x\| = \|x\|^2$ for all $x \in A$.

A *unitary 0-category* is a *unitary algebra*, i.e., a finite-dimensional C*-algebra. A unitary 0-category is equivalently a C*-0-category.

5.1.1. H*-*algebras.* An H*-*algebra*[38] (A, Tr_A) consists of a unitary algebra A and a faithful positive trace $\mathrm{Tr}\colon A \to \mathbb{C}$.[39][40] Thus A is simultaneously a unitary algebra and a Hilbert space $A = L^2(A, \mathrm{Tr}_A)$ with inner product $\langle a|b\rangle := \mathrm{Tr}_A(a^\dagger b)$.

5.1.2. H*-*modules.* H*-algebras are the "correct" objects to act on Hilbert spaces: **TODO:**

5.2. **Dagger 1-categories.** A *dagger 1-category* $(\mathcal{C}, \dagger)$ is a linear 1-category $\mathcal{C}$ with a conjugate-linear functor $\dagger\colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ that is strictly anti-involutive on 1-cells and the identity on 0-cells. Thus a dagger 1-category $\mathcal{C}$ is a vertical categorification of a $*$-algebra.

5.2.1. C*-*1-categories.* A C*-*1-category* $\mathcal{C}$ is a vertical categorification of a C*-algebra, i.e., a dagger 1-category $\mathcal{C}$ such that (C*1) for all $f \in \mathcal{C}(a \to b)$, there is a $g \in \mathcal{C}(a \to a)$ with $f^\dagger \circ f = g^\dagger \circ g$,[41] and (C*2) for each $f, g \in \mathcal{C}(a \to b)$, the function $\|-\| : \mathcal{C}(a \to b) \to [0, \infty]$ given by $\|f\|^2 := \sup\left\{|\lambda| \geq 0 \;\middle|\; f^\dagger \circ f - \lambda \mathrm{id}_a \text{ is not invertible}\right\}$ is a complete submultiplicative[42] norm satisfying (C*) $\left\|f^\dagger \circ f\right\| = \|f\|^2$ for all $f \in \mathcal{C}(a \to b)$.[43] A W*-*1-category* $\mathcal{C}$ is a C*-1-category whose hom Banach spaces have a predual.

5.2.2. *Unitary 1-categories.* A *unitary 1-category* is a dagger category $\mathcal{C}$ such that every n-fold linking algebra is unitary.[44] A unitary category is called *finite* if there is a global bound on the dimensions of the centers of all linking algebras.

5.2.3. *Pre-2-Hilbert spaces.* A *pre-2-Hilbert space* $(\mathcal{C}, \mathrm{Tr}^{\mathcal{C}})$ is a finite unitary 1-category $\mathcal{C}$ equipped with a *unitary trace* $\mathrm{Tr}^{\mathcal{C}}$, which is a collection of linear maps $\mathrm{Tr}^{\mathcal{C}}_c\colon \mathrm{End}_{\mathcal{C}}(c) \to \mathcal{C}$ for each $c \in \mathcal{C}$ such that (i) $\mathrm{Tr}^{\mathcal{C}}_c(gf) = \mathrm{Tr}^{\mathcal{C}}_d(fg)$ for all $f \in \mathcal{C}(c \to d)$ and $g \in \mathcal{C}(d \to c)$ and (ii) the sesquilinear form $\langle f|g\rangle_{c \to d} := \mathrm{Tr}^{\mathcal{C}}_c(f^\dagger g)$ on $\mathcal{C}(c \to d)$ is positive-definite.[45]

An *isometry* between pre-2-Hilbert spaces is a fully faithful $\dagger$-functor strictly preserving the unitary trace. Two pre-2-Hilbert spaces are *isometrically equivalent* if there is an essentially surjective isometry between them.[46]

5.2.4. *2-Hilbert spaces.* A *2-Hilbert space* is a Cauchy complete[47] pre-2-Hilbert space. In this case, for simple $s \in \mathcal{C}$, we define its *quantum dimension* $d_s := \mathrm{Tr}^{\mathcal{C}}_s(\mathrm{id}_s)$.

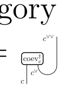
A $\dagger$-equivalence $F\colon (\mathcal{C}, \mathrm{Tr}^{\mathcal{C}}) \to (\mathcal{D}, \mathrm{Tr}^{\mathcal{D}})$ between 2-Hilbert spaces is isometric if and only if it preserves quantum dimensions, i.e., for all $c \in \mathrm{Irr}(\mathcal{C})$, $d_{F(c)} = d_c$.

5.3. **Dagger monoidal 1-categories.** A *dagger monoidal 1-category* $(\mathcal{C}, \dagger, \otimes, 1; \lambda, \rho)$ is a dagger 1-category $(\mathcal{C}, \dagger)$ with a monoidal structure such that $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a linear †-1-functor and all associators and unitors are unitary.

5.3.1. C*-*monoidal 1-categories.* A C*-*monoidal 1-category* is a unitary category $\mathcal{C}$ equipped with a dagger monoidal structure.[48]

5.3.2. *Unitary multitensor 1-categories.* A *unitary (multi)tensor* (resp. *unitary multi(fusion)*) 1-category is a multi(tensor) (resp. multi(fusion)) 1-category $\mathcal{C}$ whose monoidal structure is C*.

A C*-*monoidal 1-category* is a unitary 1-category $\mathcal{C}$ with a monoidal structure such that $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a †-functor and all associators and unitors are unitary.[49] A unitary (multi)tensor (resp. unitary (multi)fusion) category is a C*-monoidal 1-category $\mathcal{C}$ that is (multi)tensor (resp. (multi)fusion).

5.3.3. *Unitary dual functors.* A *unitary dual 1-functor* on a unitary multitensor category is a dual functor $\vee \colon \mathcal{C} \to \mathcal{C}^{\otimes\mathrm{op},\circ\mathrm{op}}$ for which the canonical maps $c \to c^{\vee\vee}$ given by $\varphi_c :=$ 
assemble into a monoidal natural isomorphism $\varphi \colon \mathrm{id}_{\mathcal{C}} \Rightarrow \vee \circ \vee$.[50] By **TODO: cite Pen20, Theorem A**, for a unitary multitensor category $\mathcal{C}$, there are canonical bijections between (1) pseudounitary pivotal structures up to monoidal natural isomorphism (that is then necessarily unique), (2) unitary dual functors up to unitary monoidal natural isomorphism (that is then necessarily unique), and (3) groupoid homomorphisms $\mathcal{U}_{\mathcal{C}} \to \mathbb{R}_{>0}$.[51]

5.4. **Dagger 2-categories.** A *dagger 2-category* is a 2-category $\mathfrak{C}$ with a conjugate-linear 2-functor $\dagger \colon \mathfrak{C} \to \mathfrak{C}^{2\mathrm{op}}$ that is strictly anti-involutive on 2-cells, the identity on 0- and 1-cells, preserved under 1-composition 1-functors, and with respect to which unitors and associators are unitary.

A *pre-unitary 2-category* is a locally unitary rigid †-2-category $\mathfrak{C}$. A *unitary 2-category* is a unitarily Cauchy complete pre-unitary 2-category.[52] A unitary 2-category is called *finite* if it has only finite many unitary equivalence classes of simple objects, and every hom unitary category is finitely semisimple.[53]

We then call $\mathfrak{C}$ (resp. W*) a C*–*2-category* (resp. W*-2-category) if it is also locally C* (resp. if (W*1) it is locally W* and (W*2) 1-composition ∘ is weak* continuous in each slot).

5.5. **Dagger 3-categories.** A *dagger 3-category* is **TODO: see Gio's paper**

A C*-*3-category* is a 3-category whose hom 2-categories are C*-2-categories and such that the underlying coherence 2-functors, 2-natural transformations, and 2-modifications are †-2-functors, †-2-natural transformations, and unitary 2-modifications respectively.

A W*-*3-category* is a C*-3-category such whose hom 2-categories are W*-2-categories and such that the 1-composition †-2-functor is weak* continuous in each slot.

6. Dagger functors

6.1. **Dagger 0-functors.** TODO:

6.2. **Dagger 1-functors.** A †-functor is a 1-functor that strictly preserves †.

6.3. **Dagger 2-functors.** TODO:

6.4. **Dagger 3-functors.** A *†-3-functor* $F\colon \mathscr{C} \to \mathfrak{D}$ between C*-3-categories s a 3-functor that is locally a †-2-functor and whose underlying coherence 2-natural transformations and 2-modifications are †-natural transformations and unitary 2-modifications respectively.

7. Strictness for dagger stuff

7.0.1. *Strict* C*-*2-categories.* By [here, Thm. 2.9], every C*- (resp. W*-) 2-category is equivalent to a C*- (resp. W*-) 2-category with strict underlying 2-category.

7.0.2. *Semistrict* C*-*3-categories.* By [here, Thm. 3.25], every C*- (resp. W*-) 3-category is equivalent to a C*- (resp. W*-) Gray-3-category. A C*-Gray-category is a C*-Gray-enriched 1-category. The exact same statement holds with each "C*" replaced with W*.

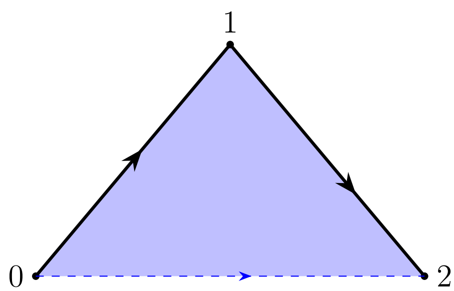
8. Homotopy coherent higher categories

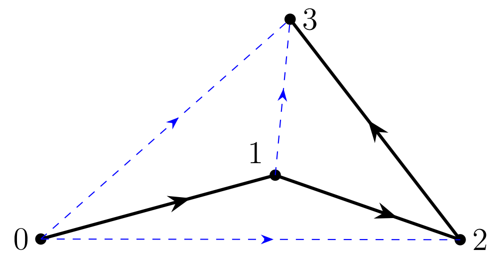
8.1. **∞-groupoids.** The *simplex category* Δ is the category of (nonempty) totally ordered finite sets and order-preserving set-functions between them. Thus the isomorphism classes of objects of Δ consist of $[n] \coloneqq \{0 < 1 < \cdots < n\}$ for nonnegative integers n. A *simplicial set* is a functor $X\colon \Delta^{\mathrm{op}} \to \mathsf{Set}$. The *standard n-simplex* is the functor $\Delta[n] \coloneqq \Delta(- \to [n])$, which is a functor $\Delta^{\mathrm{op}} \to \mathsf{Set}$ and thus a simplicial set. (Note that this means $\Delta[n]$ is the simplicial set represented by the object $[n] \in \Delta$.) The *kth horn* $\Lambda_k[n]$ of the n-simplex $\Delta[n]$ is the boundary $\partial\Delta[n]$ with the kth face removed.

A simplicial set $X\colon \Delta^{\mathrm{op}} \to \mathsf{Set}$ satisfies the *Kan condition* if all horns in X (that is, all simplicial maps $\Lambda_k[n] \to X$) extend to the full simplex in the sense that the lift in the following diagram exists.

$$\begin{array}{ccc} \Lambda_k[n] & \xrightarrow{\ \forall\ } & X \\ \big\downarrow & \overset{\exists}{\nearrow} & \\ \Delta[n] & & \end{array}$$

The following diagram on the left (resp. right) shows an extension, colored blue, of the horn $\Lambda_1[2]$ (resp. $\Lambda_1[3]$), colored black, to the 2-simplex $\Delta[2]$ (resp. 3-simplex $\Delta[3]$).





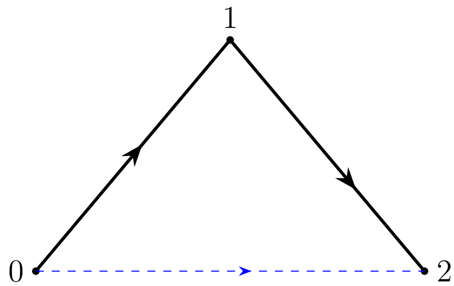
(We say the unique map $X \to \mathrm{pt}$ is a *Kan fibration*, or that X is *fibrant*, if X satisfies the Kan condition.) An *∞-groupoid* (AKA *$(\infty, 0)$-category* or *Kan complex*) is a simplicial set satisfying the Kan condition.

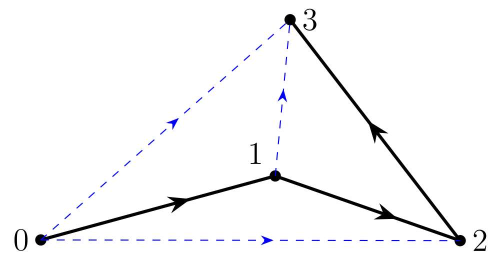
Given a topological space X, we can define an ∞-groupoid $\Pi_\infty(X)$, called the *fundamental ∞-groupoid* of X, to be the simplicial set $\{X_n\}_{n\geq 0}$ with X_0 the points in X, X_1 the paths between points, X_2 the homotopies of those paths, X_3 the homotopies between those homotopies, and so on. Conversely, given an ∞-groupoid $\{X_n\}_{n\geq 0}$, there is a functor $|-|$, called the *geometric realization functor*, from simplicial sets to topological spaces that gives a space $X \coloneqq |\{X_n\}_{n\geq 0}|$.

The *homotopy hypothesis* asserts that there is a bijective correspondence between ∞-

groupoids and topological spaces up to homotopy. More precisely, the bijection is

$$\begin{aligned} \{\text{topological spaces}\} &\longleftrightarrow \{\infty\text{-groupoids}\}, \\ X &\longmapsto \Pi_\infty(X), \\ |\{X_n\}_{n\geq 0}| &\longleftarrow X. \end{aligned} \tag{8.1.1}$$





In any notion of n-category, a k-morphism is a morphism between $(k-1)$-morphisms, which themselves are morphisms between $(k-2)$-morphisms, and so on, all the way down objects. This procedure ends with exactly two objects, namely a source object and a target object. Thus, any k-morphism can be constructed inductively by starting with the source and target objects, selecting the 1-morphisms connecting them, then the 2-morphisms between those, followed by the 3-morphisms between the 2-morphisms, and so on, until level k. We can actually just skip steps to

A *simplicial space* is a simplicial object in Top, i.e., a functor $X\colon \Delta^{\mathrm{op}} \to \mathsf{Top}$. A simplicial space X satisfies the *Segal condition* if, up to homotopy, the k-cells of X are precisely the composites of composable sequences of k morphisms. For example, for any two composable morphisms, say given as in the following figure, there is a contractible choice of dashed filler that makes the triangle commute up to homotopy [see above two simplicial sets figures]. By considering iterates of homotopy pullbacks of X_1 (morphisms) over X_0 (objects), that is, by choosing composable morphisms for each sequence of n-objects, we find that this is equivalent to the following.

Let's show that a Segal space is still not enough to satisfy our desideratum for $(\infty, 1)$-categories. To see why, suppose it did, and let $\mathcal{C}$ be a complete Segal space. A point in our desideratum **TODO: make desideratum and ensure this is one of the items therein** says that the classifying space of the underlying groupoid of $\mathcal{C}$ should admit an "inverse"; that is, the classifying

[From here] For a Segal space X, the *objects* are elements of $\mathrm{Obj}(X) \coloneqq X_0$, while its *1-morphisms* are elements of the *mapping space* $\mathrm{Map}_X(x, y)$ for some objects $x, y \in X_0$, which is defined as the fiber of the map $(d_0, d_1)\colon X_1 \to X_0 \times X_0$ over (x, y) sending an element $f \in X_1$ to the pair structure. The *identity morphism* of an object $x \in X$ given by the image of x under the degeneracy map $s_0\colon X_0 \to X_1$.

Two 1-morphisms f and g in X_0 are *homotopic*, written $f \simeq g$, if they lie in the same connected component of X_0.

Given $f \in \mathrm{Map}_Z(x, y)$ and $g \in \mathrm{Map}_X(y, z)$, there is a composite $g \circ f \in \mathrm{Map}_X(x, z)$, and this notion of composition is associative up to homotopy by the Segal condition. The *homotopy category* $\mathsf{Ho}(X)$ of X has as objects the set $\mathrm{Obj}(X)$ and as morphisms between any two objects x and y, the set $\mathrm{Map}_{\mathsf{Ho}(X)}(x, y) \coloneqq \pi_0 \mathrm{Map}_X(x, y)$, the set of connected components (or, by our definition of homotopy in X_0, equivalently, homotopy classes) of X_0.

Finally, a map g in $\mathrm{Map}_X(x, y)_0$ is a *homotopy equivalence* if there exist maps $f, h \in \mathrm{Map}_X(y, x)_0$ such that $g \circ f \sim \mathrm{id}_y$ and $h \circ g \sim \mathrm{id}_x$. Any map in the same component as a homotopy equivalence is itself a homotopy equivalence. Therefore we can define the space X_{hoequiv} to be the subspace of X_1 given by the components whose 0-simplices are homotopy equivalences.

Xe then note that the degeneracy map $s_0 \colon X_0 \to X_1$ factors through X_{hoequiv} since for any object x the map $s_0(x) = \mathrm{id}_x$ is a homotopy equivalence. Therefore, we have the following definition:

Definition 8.1.2. A *complete Segal space* is a Segal space X for which the map $s_0 \colon X_0 \to X_{\mathrm{hoequiv}}$ is a weak equivalence of simplicial sets.

We can now consider some particular kinds of maps between Segal spaces.

Definition 8.1.3. A map $f \colon U \to V$ of Segal spaces is a *DK-equivalence* if:

(i) For any pair of objects $x, y \in U_0$, the induced map

$$\mathrm{Map}_U(x, y) \to \mathrm{Map}_V(fx, fy)$$

is a weak equivalence of simplicial sets.

(ii) The induced map $\mathsf{Ho}(f) \colon \mathsf{Ho}(U) \to \mathsf{Ho}(V)$ is an equivalence of categories.

We are now able to describe the important features of the complete Segal space model category structure.

Theorem 8.1.4. *There is a model structure $\mathcal{M}\mathrm{s}\mathsf{Top}$ on the category of simplicial spaces such that:*

(i) *The weak equivalences between Segal spaces are the DK-equivalences.*

(ii) *The cofibrations are the monomorphisms.*

(iii) *The fibrant objects are the complete Segal spaces.*

What makes the model category $\mathcal{M}\mathrm{s}\mathsf{Top}$ so nice to work with is the fact that the weak equivalences between the fibrant objects, the complete Segal spaces, are easy to identify.

Proposition 8.1.5. *A map $f \colon U \to V$ between complete Segal spaces is a DK-equivalence if and only if it is a levelwise weak equivalence.*

8.2. A_n**-algebras.**

8.3. E_n**-algebras.**

8.4. **Coherence.** [Lurie's HTT, §1.2.6]: A *homotopy coherent diagram* in $\mathcal{C}$ is a functor $F\colon \mathcal{J} \to \mathsf{Ho}(\mathcal{C})$ together with all of the extra data (read: choices) that would be available if we were able to lift F to a functor $\widetilde{F}\colon \mathcal{J} \to \mathcal{C}$.

9. Appendix

9.1. **Disklike n-categories.**

(i) [Here, Axm. 6.1.1] (Morphisms): For each $0 \leq k \leq n$, we have a functor $\mathcal{X}_k$ from the category of k–balls and homeomorphisms to the category of sets and bijections.

(ii) [Here, Lem. 6.1.2] (Boundaries of morphisms) For each $1 \leq k \leq n$, we have a functor $\underset{\rightarrow}{\mathcal{X}}_{k-1}$ from the category of $(k-1)$-spheres to the category of sets and bijections.

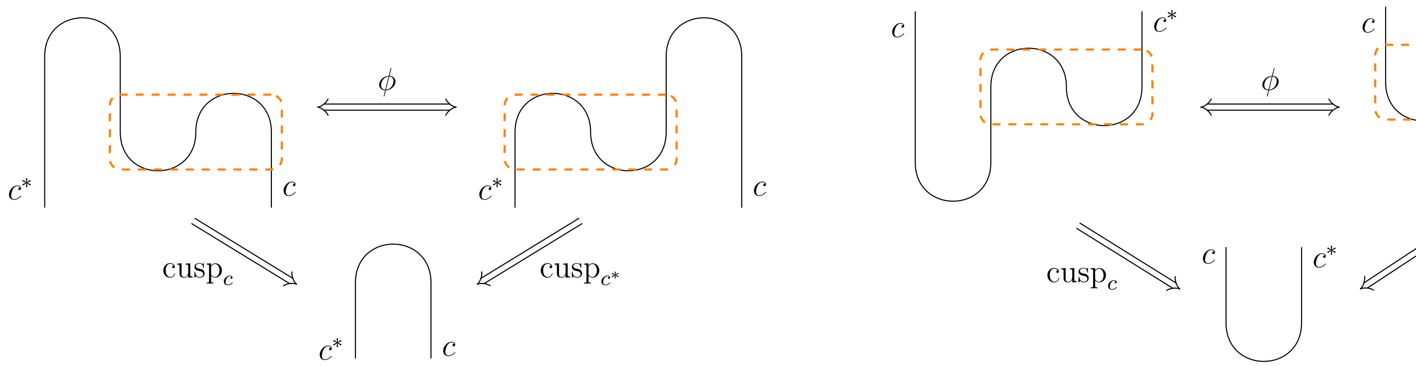
(iii) [Here, Axm. 6.1.3] (Boundaries) For k–balls X, we have maps of sets $\partial\colon \mathcal{X}_k(X) \to \underset{\rightarrow}{\mathcal{X}}_{k-1}(\partial X)$ assembling into a natural transformation.

9.2. **Alternate formulation of multitensor categories.** The following presents multitensor categories according to [EGNO15].

9.3. **Linear abelian categories.** A category is *additive* if it has a zero object and finite biproducts (often called *direct sums*).[54] An *abelian category* is an additive category with all kernels and cokernels and such that monomorphisms (resp. epimorphisms) are kernels (resp. cokernels).[55][56] A *subobject* of an object x in an abelian category $\mathcal{C}$ is a monomorphism $a \hookrightarrow x$. A *quotient object* of x is an object z together with an epimorphism $x \twoheadrightarrow z$. For a subobject $a \hookrightarrow x$, define the *quotient object* x/a to be the cokernel of the monomorphism $a \twoheadrightarrow x$. We call x *simple* if it admits no nontrivial subobjects.[57] Let $\mathrm{Irr}(\mathcal{C})$ denote the isomorphism classes of simple objects in $\mathcal{C}$.

A *linear abelian* category is an abelian category whose Ab-enrichment is lifted to a Vec-enrichment. A linear abelian category $\mathcal{C}$ is *locally finite* if hom spaces are finite-dimensional and objects have finite length.[58] A *multitensor category* is a locally finite linear abelian rigid semisimple monoidal category $(\mathcal{C}, \otimes, 1)$ such that $\otimes$ is bilinear.[59][60] A *tensor category* is a multitensor category with simple unit 1. A *(multi)fusion category* is a finite semisimple (multi)tensor category.

9.4. **Cusp-flip (swallowtail) equations.** By "coherent" here we mean that cusp and cocusp satisfy the *cusp flip* (or *swallowtail*) equations, which assert the following diagrams commute.



9.5. **Fusion 0-categories.**

9.5.1. $\mathbb{Z}_+$*-rings.* Where $\mathbb{Z}_+$ denotes the semiring of nonnegative integers $\mathbb{Z}_{\geq 1}$ under addition, a $\mathbb{Z}_+$*-ring* (S, A) consists of a ring A whose underlying abelian group is a finitely generated free $\mathbb{Z}$-module with basis $S = \{s_i\}_{i \in I}$ (whose elements are called *simple*) for which $s_i \cdot s_j$ is a nonnegative (i.e., $\mathbb{Z}_+$-)linear combination of all the $s_k \in S$. Thus (left, say) multiplication by s_i is a matrix N_{s_i} with $\mathbb{Z}_+$-valued entries, so by the Frobenius–Perron theorem it has a positive real eigenvalue $d_{s_i} =: \mathrm{FPdim}(s_i)$ of maximal spectral radius, called the *Frobenius–Perron dimension* of s_i. We define $\mathrm{FPdim}(\mathcal{C}) := \sum_{c \in \mathrm{Irr}(\mathcal{C})} \mathrm{FPdim}(c)^2$.

We call a $\mathbb{Z}_+$-ring (S, A) a *multifusion ring* if $1_A \in S$ and for every simple $s_i \in S = \{s_1, \ldots, s_n\}$ there exists a unique simple $s_i^* \in S$ such that

$$c_{s_i, s_j}^1 = \begin{cases} 1 & \text{if } s_j = s_i^*, \\ 0 & \text{otherwise} \end{cases}$$

and the quantity $c_{s_i, s_j}^{s_k^*}$ is invariant under cyclic permutations of s_i, s_j, and s_k.[61] If $1 \in S$, then (S, A) is called a *fusion ring*.

References

[EGNO15] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor Categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2015.

[Gur13] Nick Gurski. *Coherence in Tricategories*, volume 201 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013.

[JY21] Niles Johnson and Donald Yau. *2-dimensional categories*. Oxford University Press, 2021.

Notes

1. The non-abelian 4-cocycle is the commutativity of the Stasheff associahedron K_5 that describes ways to move parentheses from $(((-\boxtimes-)\boxtimes-)\boxtimes-)\boxtimes-$ to $-\boxtimes(-\boxtimes(-\boxtimes(-\boxtimes-)))$. To read the cited diagram, read juxtaposition as $\boxtimes$ and replace a with α.
2. The left and right normalization conditions compare the ways to move parentheses from $(-\boxtimes-)\boxtimes-$ to $-\boxtimes(-\boxtimes-)$ by using only $(\alpha,\lambda,\rho;\pi,m,\ell)$ and $(\alpha,\lambda,\rho;\pi,m,\rho)$ respectively. To read the cited diagram, read juxtaposition as $\boxtimes$, replace a with α, and make the swaps $\mu\leftrightarrow m$, $\rho\leftrightarrow r$, and $\lambda\leftrightarrow\ell$.
3. If the 1-unitor and 1-compositor are not required to be invertible, we call F a *lax 2-functor*. If we have a lax 2-functor except the domain and codomain of both μ and ν are reversed, we call F a *oplax 2-functor*.
4. If the invertibility condition on the 2-cell η is relaxed, i.e., if the (η_a's and the) η_f's need not be invertible, then we instead call η a *lax 2-natural transformation*. If η is a lax 2-natural transformation but with the direction of the 2-cells (η_f's) reversed, we instead call η a *oplax 2-natural transformation*.
5. 2-cells of this type only exist because $F=G$ on 0-cells.
6. That is, a local bijection, meaning a bijection on hom sets.
7. A 2-natural equivalence is a 1-equivalence in the 2-category of 2-functors, 2-natural transformations, and 2-modifications.
8. A 2-functor is fully faithful on 0- and 1-cells if it is an equivalence on the hom 1-functors, meaning a local equivalence.
9. As an exercise, show that for a general linear monoidal category $\mathcal{C}$, we can ask for at most 2 out of 3 of the properties strict, skeletal, and Cauchy complete.
10. From [here, Lemma 2.2]
11. Can you generalize the previous footnote to monoidal 2-categories?
12. This is from here, Notation A.1.
13. For a $\mathcal{V}$-enriched 1-category $\mathcal{C}$, the *underlying category* $\mathcal{C}^{\mathcal{V}}$ is the 1-category whose objects are the same objects as $\mathcal{C}$ and whose morphisms are the 1-cells of $\mathcal{C}$ as defined above. For $f\in\mathcal{C}(a\to b)$ and $g\in\mathcal{C}(b\to c)$, which by above means $f\in\mathcal{V}(1_{\mathcal{V}}\to\mathcal{C}(a\to b))$ and $g\in\mathcal{V}(1_{\mathcal{V}}\to\mathcal{C}(b\to c))$, their composition in $\mathcal{C}^{\mathcal{V}}$, $g\circ f\in\mathcal{C}(a\to c)$ (so $g\circ f\in\mathcal{V}(1_{\mathcal{V}}\to\mathcal{C}(a\to c))$) and the identity morphisms in $\mathcal{C}^{\mathcal{V}}$ are defined in the obvious way.
14. Here *locally linear* means the hom sets are finite-dimensional complex vector spaces.
15. We assume 1-functors of linear 1-categories are linear.
16. The columns correspond to the domain and the rows correspond to the codomain, similar to how a matrix acts as a linear operator.
17. When $\mathcal{C}$ has a dagger structure, $L(a,b)$ is a $*$-algebra with involution $*$ being the †-transpose. Similarly, we can define the linking algebra $L(a_1,\ldots,a_n)$ for n objects $a_1,\ldots,a_n\in\mathcal{C}$.

18. That is, there is a $K > 0$ such that for any linking algebra $\mathcal{L} = \mathcal{L}(a_1, \ldots, a_n)$, $\dim(\mathrm{End}(1_{Z(\mathcal{L})})) < K$.

19. A 1-functor $F \colon \mathcal{C} \to \mathcal{D}$ is *dominant* if for every object $d \in \mathcal{D}$, there is an object $c \in \mathcal{C}$ and morphisms $d \xrightarrow{i} F(c) \xrightarrow{p} d$ such that $p \circ i = \mathrm{id}_d$. That is, F is dominant if each $F(c)$ admits some 1-idempotent $e \in \mathcal{D}(F(c) \to F(c))$ that splits. A fully faithful functor between 1-idempotent complete categories is an equivalence if and only if it is dominant.

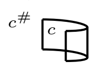
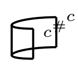
20. Henceforth, by "algebra" here we will always mean "unital algebra".

21. To obtain the definition of a (counital) *coalgebra*, dualize the definition of a (unital) algebra.

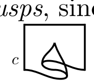
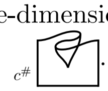
22. Equivalently, a condensation algebra splits if there is an adjunction ${}_aF_b \dashv {}_bU_a$ of 1-morphisms $(F, U, \mathrm{ev} \colon U \otimes F \Rightarrow 1_b, \mathrm{coev} \colon 1_a \Rightarrow F \otimes U)$ whose associated monad[62] is isomorphic to A as algebras in $\mathrm{End}_{\mathfrak{C}}(a)$.

23. Equivalently, a linear 1-category is semisimple if it is Cauchy complete and the endomorphism algebra of every object is semisimple.

24. Here we are adopting the desideratum from **TODO: Dualizable Tensor Categories**, which is that "in reasonable settings where something could be called either a dual or an adjoint, the left dual should be the left adjoint".

25. Here ev and coev are called *folds*, since in the three-dimensional diagrammatical calculus for monoidal 2-categories, they look like ev =  and coev =  .

26. By "coherent" here we mean that cusp and cocusp satisfy the *cusp flip* (or *swallowtail*) equations.

27. Here cusp and cocusp are called *cusps*, since in the three-dimensional diagrammatical calculus for monoidal 2-categories, they look like cusp =  and cocusp = .

28. Here *adjoint* is used to mean dual in the usual sense. The reason for the difference of terminology is that it is typical to reserve "dual" for 0-cells and "adjoint" for k-cells for $k \geq 1$.

29. Equivalently, a monoidal 2-category is rigid if it is locally rigid and its object admit left and right duals.

30. A multitensor category is *indecomposable* if it is not equivalent to a direct sum of nonzero multitensor categories.

31. In this setting, by [here, Prop. 1.2.14], the unit object $\mathbf{1}$ is simple if and only if its identity (1-)morphism $1_{\mathbf{1}}$ is simple.

32. **TODO: I think? Still TBD.**

33. Really, we only require here that 0-cells admit duals.

34. Really, we only require here that 1-cells admit adjoints.

35. You can think about what this means if you'd like, but here, Def. 2.2.3 just uses the semistrictification for monoidal 2-categories to require that $\vee$ and $\mathrm{ev}, \mathrm{coev}$ strictly preserve $\boxtimes$.

36. Recall that a *torsor* (AKA *principal homogeneous space*) for a group G is a nonempty free and transitive

G-set X. A *pivotal 1-category* is a multitensor category equipped with a pivotal structure. Thus, to say the set of pivotal structures for a pivotal multitensor category $(\mathcal{C}, \varphi)$ is a torsor for G means that for any other pivotal structure φ', $g \cdot \varphi = \varphi'$ for some $g \in G$, which means the set of pivotal structures for $\mathcal{C}$ is $\{g \cdot \varphi \mid g \in G\}$.

37. Informally, this says "$\vee \delta \varphi^{\mathcal{D}} F = \delta \vee F \varphi^{\mathcal{C}}$".

38. Recall that given a faithful positive linear functional φ on a (finite-dimensional) unitary algebra A, we get the GNS Hilbert space $L^2(A, \varphi) = A$ with inner product $\langle a|b\rangle := \varphi\left(a^\dagger b\right)$. If φ is also tracial, and we remember the algebra structure of A, we get the notion of an H*-algebra.

39. We call a trace $\mathrm{Tr}_A \colon A \to \mathbb{C}$ *faithful* if for all $a \in A$, $\mathrm{Tr}_A(a^\dagger a) = 0$ implies $a = 0$, and *positive* if for all $a \in A$, $\mathrm{Tr}(a^* a) \geq 0$.

40. We often identify H*-algebras (A, Tr_A) with their GNS Hilbert space $L^2(A, \mathrm{Tr}_A)$.

41. That is, positive morphisms are spectrally positive.

42. That is, for all $f \in \mathcal{C}(a \to b)$ and $g \in \mathcal{C}(b \to c)$, $\|g \circ f\| \leq \|g\| \cdot \|f\|$.

43. If $\mathcal{C}$ admits direct sums, then (C*1) and (C*2) together are equivalent to the assertion (C*) $\mathrm{End}(a \oplus b)$ is a C*-algebra for all $a, b \in \mathcal{C}$.

44. From unitarity of the linking algebras, unitary categories are C* and W* categories. A unitary category is semisimple if and only if it is Cauchy complete. A unitary category admits direct sums if and only if it admits orthogonal direct sums and is idempotent complete if and only if it is projection complete, i.e., all orthogonal projections split orthogonally. We thus define the *Cauchy completion* of a unitary category as the projection completion of the orthogonal additive envelope and say that such categories are *unitarily Cauchy complete*.

45. Observe that the inner products $\langle f|g\rangle_{a\to b}$ on the Hilbert spaces $\mathcal{C}(a \to b)$ satisfy $\left\langle g\middle|hg^\dagger\right\rangle_{b\to c} = \langle gf|h\rangle_{a\to c} = \left\langle f\middle|g^\dagger h\right\rangle_{a\to b}$ for all $f \in \mathcal{C}(a \to b)$, $g \in \mathcal{C}(b \to c)$, and $h \in \mathcal{C}(a \to c)$.

46. Note that this definition is equivalent to having isometries both ways together with unitary natural isomorphisms from their composites to the appropriate identity functors.

47. Or equivalently, in this case, semisimple.

48. Just as with the adjective tensor, we reserve the desirable adjective "unitary" for rigid C*-monoidal categories. (It would make no difference at this categorical level to use "unitary monoidal" category instead of "C*-monoidal" category, but when we get to 2-categories we will want the adjective "unitary" to include having adjoints for 1-morphisms.)

49. In other words, a C*-monoidal 1-category is a unitary 1-category with a *dagger monoidal* structure.

50. Equivalently, a $\vee$ is a unitary dual functor if its canonical tensorators are unitary and $\vee\dagger = \dagger\vee$.

51. Here $\mathcal{U}_{\mathcal{C}}$ denotes the universal grading groupoid of $\mathcal{C}$.

52. See here for more details.

53. See here for more details.

54. A functor between additive categories is *additive* if it preserves the zero object and biproducts up to isomorphism.

55. For any category $\mathcal{C}$, a *kernel* of a morphism $f \in \mathcal{C}(a \to b)$ is the universal morphism $k \in \mathcal{C}(\ker(f) \to a)$ that f precomposes to 0. A *cokernel* is the kernel in $\mathcal{C}^{\text{op}}$.

56. Functors of abelian categories are assumed additive.

57. For linear categories, this is equivalent to the endomorphism space of each object being isomorphic to the underlying field.

58. That is, objects admit finite composition series. A *composition series* of an object $x \in \mathcal{C}$ is a filtration $0 = x_0 \hookrightarrow x_0 \hookrightarrow \cdots \hookrightarrow x_n = x$ such that $x_i/x_{i-1} \overset{\text{def}}{=} \text{coker}(x_{i-1} \hookrightarrow x_i)$ is simple for all i.

59. By [EGNO15, Prop. 4.2.1], the $\otimes$ in a multitensor category is exact in both slots. Thus a multitensor category is a rigid *multiring* category.

60. A multitensor category is *indecomposable* if it is not equivalent to a direct sum of nonzero multitensor categories.

61. One can show $*$ linearly extends from an involution on S to an involution $*\colon A \to A$.

62. An adjunction of 1-morphisms ${}_aF_b \dashv {}_bU_a$ in $\mathfrak{C}$ with counit $\text{ev}\colon U \otimes F \Rightarrow 1_b$ and unit $\text{coev}\colon 1_a \Rightarrow F \otimes U$ has an associated monad (monoid object in $\text{End}_{\mathfrak{C}}(a)$) given by $({}_aF \otimes_b U_a, \text{id}_F \otimes \text{ev} \otimes \text{id}_U\colon F \otimes U \otimes F \otimes U \Rightarrow F \otimes U, \text{coev}\colon 1_a \Rightarrow F \otimes U)$. Whenever $X \dashv Y$ is a splitting for A, we say $X \dashv Y$ *splits* A, or simply that A *splits*.